Little survey on large fields – Old & New –

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Abstract. The large fields were introduced by the author in [59] and subsequently acquired several other names. This little survey includes earlier and new developments, and at the end of each section we mention a few open questions.

2010 Mathematics Subject Classification. Primary 12E, 12F, 12G, 12J. Secondary 12E30, 12F10, 12G99.

Keywords. Large fields, ultraproducts, PAC, pseudo closed fields, Henselian pairs, elementary equivalence, algebraic varieties, rational points, function fields, (inverse) Galois theory, embedding problems, model theory, rational connectedness, extremal fields.

Introduction

The notion of large field was introduced in Pop [59] and proved to be the "right class" of fields over which one can do a lot of interesting mathematics, like (inverse) Galois theory, see Colliot-Thélène [7], Moret-Bailly [45], Pop [59], [61], the survey article Harbater [30], study torsors of finite groups Moret-Bailly [46], study rationally connected varieties Kollár [37], study the elementary theory of function fields Koenigsmann [35], Poonen–Pop [55], characterize extremal valued fields as introduced by Ershov [12], see Azgin–Kuhlmann–Pop [1], etc. Maybe that is why the "large fields" acquired several other names —google it: épais, fertile, weite Körper, ample, anti-Mordellic. Last but not least, see Jarden's book [34] for more about large fields (which he calls "ample fields" in his book [34]), and Kuhlmann [41] for relations between large fields and local uniformization (à la Zariski).

Definition. A field k is called a large field, if for every irreducible k-curve C the following holds: If C has a k-rational smooth point, then C has infinitely many k-rational points.

As we will see below, the class of large fields is an elementary class, more precisely being a large field can be expressed by a countable set of axioms in the language of fields. In particular, given a family of large fields $(k_i)_{i\in I}$ and an ultrafilter $\mathcal{U}$ on I, the ultraproduct $\prod_i k_i/\mathcal{U}$ is a large field, and if k is a field with $k^I/\mathcal{U}$ a large field, then k is a large field. Further, every algebraic extension $l|k$ of a large field k is a large field. See e.g., [5] and/or [18] for basic facts and more on ultraproducts and existentially closed extensions of rings, fields, etc. Further, the class of large fields is quite rich, and it contains –among other things, the fields complete with respect to non-trivial absolute values, the fields which are Henselian with respect to a non-trivial valuation, the fields which are pseudo closed with respect to families of orderings and valuations, the quotient fields of generalized Henselian domains, and much more.

PART I: **Basic Facts**

We fix notations to be used throughout the paper and recall a few basic facts as follows: Let k be an arbitrary field. A k-variety is any separated reduced k-scheme of finite type, and a k-curve is a k-variety of dimension one. For every k-variety X and every field extension $l|k$ we denote by $X(l)$ the l-rational points of X, hence $X(k)$ is the set of k-rational points of X. Given a k-variety X, we denote by $X_{\rm sm}$ and $X_{\rm reg}$ the smooth, respectively regular, locus of X. We recall that $X_{\rm sm}$ and $X_{\rm reg}$ are open k-subvarieties

*The author is grateful to the Hausdorff Institute for Mathematics HIM Bonn, Germany, for hospitality during the writing of this paper. The author was supported by the NSF Research Grant DMS-1101397.

of X, and that $X_{\rm sm}$ is non-empty if and only if the function field $k(X)$ of X is separably generated over k, see e.g. Matsumura [44], Ch. 9, especially section 26. Further, by Lemma 1 in Ch. 10, section 28 of loc.cit., it follows that $X_{\rm sm} \subseteq X_{\rm reg}$, and second, if $x \in X_{\rm reg}$ has residue field $\kappa(x)$ separably generated over k, then $x \in X_{\rm sm}$. In particular, a k-rational point $x \in X$ *is regular if and only if it is smooth,* and therefore, $X_{\rm reg}(k) = X_{\rm sm}(k)$. (One could use the term "non-singular" instead of "smooth," but that terminology is used usually over algebraically closed base fields k, see e.g. Mumford [48], Ch. III, § 4).

Fact/Notations I. Let X be an irreducible k-variety of dimension d, and $x \in X_{\rm reg}(k) = X_{\rm sm}(k)$ be a k-rational regular, thus smooth, point of X. Then one has:

1) Let $(t_1, \ldots, t_d)$ be a system of local parameters at x. Then x has a basis of affine open smooth neighborhoods U such that $t_1, \ldots, t_d \in \Gamma(U, \mathcal{O}_X)$ and U is a complete intersection as follows:

$$U = \operatorname{Spec} k[X_1, \ldots, X_{d+m}]/(f_1, \ldots, f_m) \subset \mathbb{A}_k^{d+m}$$

with $t_i = X_i \big(\mathrm{mod}(f_1, \ldots, f_m)\big)$, $1 \le i \le d$, and $f_j \in k[X_1, \ldots, X_{d+m}]$ of the form:

$$f_j = X_{d+j} + (\text{terms in } X_1, \ldots, X_{d+m} \text{ of total degree} > 1), \quad j = 1, \ldots, m.$$

In particular, the projection on the first d coordinates $U \hookrightarrow \mathbb{A}^{d+m} \to \mathbb{A}^d$ is defined by the k-embedding $k[X_1, \ldots, X_d] \hookrightarrow k[X_1, \ldots, , X_{d+m}] \to k[X_1, \ldots, X_{d+m}]/(f_1, \ldots, f_m)$.

2) There exists a hypersurface $X_0 := V(f) \subset \mathbb{A}^{d+1}$ which is birationally equivalent to X, say under a rational map $X \dashrightarrow X_0$ which is defined at x and maps x to $x_0 = (0, \ldots, 0)$, and f is of the form $f = X_{d+1} + \widetilde{f}$ with $\widetilde{f} \in k[X_1, \ldots, X_{d+1}]$ a polynomial with vanishing terms in degrees < 2. In particular, the image $x_0 \in X_0(k)$ of x under $X \dashrightarrow X_0$ is a smooth k-rational point of X_0.

3) Moreover, in the above notations, $X(k)$ is Zariski dense in X iff $X_0(k)$ is Zariski dense in X_0.

1. Some Examples of Large Fields

A) *The basic examples of large fields*

1) The PAC fields are large fields. Indeed, recall that a field k is called PAC (pseudo algebraically closed) if every geometrically integral k-variety X has k-rational points, i.e., $X(k)$ is non-empty. On the other hand, if X is a geometrically integral k-variety, then so is every Zariski open k-subvariety $U \subseteq X$, thus if k is a PAC field, then $U(k)$ is non-empty. We thus conclude that if k is a PAC field, and X is a geometrically integral k variety, then $X(k)$ is Zariski dense. Now let X be a k-variety with $X_{\rm sm}(k)$ non-empty. Then X is algebraically integral. (Actually, if X is a k-variety, thus integral, then $X(k)$ non-empty implies that X is geometrically integral.) Hence by the discussion above one has that $X(k)$ is Zariski dense in X, and in particular, $X(k)$ is infinite.

2) Fields which are complete with respect to non-trivial absolute values are large. Indeed, let k be such a field, and $\tau_{|\,|}$ denote the strong topology, and let X be a k-variety. If $x \in X_{\rm sm}(k)$ is a given smooth k-rational point and $(t_1, \ldots, t_d)$ is a system of local parameters at x, then in the setting from Fact/Notations I one has: By the algebraic Implicit Function Theorem, see e.g. Green–Pop–Roquette [21], it follows that for every $\epsilon > 0$ sufficiently small, $X(k)$ contains a $\tau_{|\,|}$-neighborhood $\mathcal{U}_x \subset X(k)$ which is mapped by the system of rational functions $(t_1, \ldots, t_d)$ homeomorphically onto the open polydisk $B(0, \epsilon)^d$ in $k^d = \mathbb{A}_k^d(k)$. In particular, $X(k)$ is infinite, and even Zariski dense.

   Thus $\mathbb{R}$, $\mathbb{Q}_p$, $k_i((t))$ with k_i arbitrary base field, and their finite extensions, are large fields.

3) Fields which are either real closed or Henselian with respect to non-trivial valuations are large. Indeed, let Λ be such a field, and X is a Λ-variety with $X_{\rm reg}(\Lambda)$ non-empty. Then in the setting from Fact/Notations I, one can apply the algebraic Implicit Function Theorem, see e.g. Green–Pop–Roquette [21], and get that $X_{\rm reg}(\Lambda)$ is Zariski dense, thus so is $X(\Lambda)$, and hence infinite. In particular, the real or p-adically closed fields are large, and so is the henselization $k_i(t)^{\rm h}$ of the rational function field $k_i(t)$ with respect to the t-adic valuation.

B) *Pseudo $\mathcal{K}$-closed fields*

Let k be an arbitrary field. A locality of k is an algebraic extension Λ of k which is either real closed or Henselian with respect to a non-trivial valuation. Let $\mathcal{K}$ be a set of localities of k. We say that k is pseudo $\mathcal{K}$-closed, if for every geometrically integral k-variety X the following holds: $X(k)$ is non-empty, provided $X_{\rm reg}(\Lambda)$ is non-empty for every $\Lambda \in \mathcal{K}$. We notice that if k is pseudo $\mathcal{K}$-closed, and $X_{\rm reg}(\Lambda)$ is non-empty for each $\Lambda \in \mathcal{K}$, then $X_{\rm reg}(k)$ is actually Zariski dense. Indeed, for every locality $\Lambda \in \mathcal{K}$, it follows by the discussion at A), 3) above that $X_{\rm reg}(\Lambda)$ is Zariski dense (because it was non-empty by hypothesis). Thus if $U \subset X$ is any Zariski open k-subvariety, it follows that $U_{\rm reg}(\Lambda)$ is non-empty. Therefore, $U(k)$ is non-empty by the fact that k is pseudo $\mathcal{K}$-closed. Thus we conclude that $X(k)$ is Zariski dense, and therefore, so is $X_{\rm reg}(k)$, because $X_{\rm reg} \subseteq X$ is a Zariski open dense subset. We also notice that every algebraic extension of a pseudo closed field is itself pseudo closed as follows: Let k be pseudo closed with respect to $\mathcal{K}$, and $l|k$ be an algebraic extension, say contained in some fixed algebraic closure $\overline{k}|k$ of k. We define the prolongation $\mathcal{L}$ of $\mathcal{K}$ to l to be the set of all the composita $\Lambda_{l\sigma} := l\Lambda^\sigma$ of l with all conjugates $\Lambda^\sigma \subset \overline{k}$, where $\Lambda \in \mathcal{K}$ and $\sigma \in \mathrm{Aut}(\overline{k}|k)$. Then by Pop [58], Overfield Theorem, l is pseudo $\mathcal{L}$-closed. See Pop [58], and also Jarden–Haran–Pop [26], [27], for more about this.

We claim that every pseudo $\mathcal{K}$-closed field k is actually a large field. Indeed, let X be any k-variety with $X_{\rm reg}(k)$ non-empty. Then $X_{\rm reg}(k) \subseteq X_{\rm reg}(\Lambda)$, thus $X_{\rm reg}(\Lambda)$ is non-empty for all $\Lambda \in \mathcal{K}$. Hence by the discussion above it follows that $X(k)$ is Zariski dense, hence infinite as well.

We next give two quite concrete examples of pseudo $\mathcal{K}$-closed fields, both being totally $\mathfrak{S}$-adic fields. First, for a set $\mathfrak{S}$ of orderings and non-trivial valuations of a field k, we denote by $k^{\mathfrak{S}}$ a maximal algebraic extension of k in which all the $v \in \mathfrak{S}$ are totally split, and let $\mathcal{K}^{\mathfrak{S}}$ be the prolongation of $\mathfrak{S}$ to $k^{\mathfrak{S}}$. The field extension $k^{\mathfrak{S}}|k$ is Galois, and we call $k^{\mathfrak{S}}$ the totally $\mathfrak{S}$-adic field extension of k.

Recall that an absolute value of a field k, or its place $\mathfrak{p}$, is called classical, if the completion $k_{\mathfrak{p}}$ of k at $\mathfrak{p}$ is a locally compact field. That means $k_{\mathfrak{p}}$ is a finite extension of one of the fields $\mathbb{R}$, $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. In particular, if k is a global field, i.e., a number field or a function field of a curve over a finite field, then all the non-trivial places of k are classical. But the fields k which admit classical places are quite abundant, e.g., all finitely generated infinite fields have "many" classical places. (Actually given a $\mathbb{Z}$-model X of a finitely generated infinite field k, the local ring of every closed point $x \in X$ is dominated by the valuation ring of a classical place.)

The first class of concrete examples of pseudo $\mathcal{K}$-closed fields is as follows. Let k be an arbitrary field endowed with a *finite set $\mathfrak{S}$ of classical places.* Then the following (first mentioned in an unpublished note by the author) holds, for a proof see any of the papers: Moret-Bailly [45], Green–Pop–Roquette [21], Pop [59], Appendix, or Geyer–Jarden [20].

Theorem 1.1. *In the above notations, the field $k^{\mathfrak{S}}$ is pseudo $\mathcal{K}^{\mathfrak{S}}$-closed. In particular, every geometrically integral k-variety X satisfies: $X_{\rm reg}(k_{\mathfrak{p}})$ is non-empty for all $\mathfrak{p} \in \mathfrak{S}$ iff $X_{\rm reg}(k)$ is Zariski dense.*

In particular, every algebraic extension l of $k^{\mathfrak{S}}$ is a pseudo $\mathcal{L}$-closed field, where $\mathcal{L}$ is the prolongation of $\mathfrak{S}$ to l. And by the discussion above, it follows l is large as well.

The second class of concrete examples of pseudo $\mathcal{K}$-closed fields is as follows. Let k_0 be a global field of positive characteristic (thus the function field of a curve over a finite field), $\mathfrak{S}_0$ be an incomplete set of places of k_0, i.e., a proper subset of the set of all the places of k_0. For an infinite constant extension $k|k_0$ of k_0, we let $\mathfrak{S}$ be the prolongation of $\mathfrak{S}_0$ to k. Finally let $k^{\mathfrak{S}}$ and $\mathcal{K}^{\mathfrak{S}}$ be defined as previously. Then one has the following fact proved by Tamagawa [71]:

Theorem 1.2. *In the above notations, the field $k^{\mathfrak{S}}$ is pseudo $\mathcal{K}^{\mathfrak{S}}$-closed. In particular, every geometrically integral k-variety X satisfies: $X_{\rm reg}(k_{\mathfrak{p}})$ is non-empty for all $p \in \mathfrak{S}$ iff $X_{\rm reg}(k)$ is Zariski dense.*

In particular, every algebraic extension l of $k^{\mathfrak{S}}$ is a pseudo $\mathcal{L}$-closed field, where $\mathcal{L}$ is the prolongation of $\mathfrak{S}$ to l. And by the discussion above, it follows l is large as well.

C) *The p-closed fields*

Let p be a prime number. Recall that a field k is called p-closed, if every finite extension of k has degree a power of p. In particular, the absolute Galois group G_k of k is a pro-p group, and further: If $p \neq \mathrm{char}(k)$, then k is perfect, whereas if $p = \mathrm{char}(k)$, then k might also have purely inseparable extensions. To the best of my knowledge, Colliot-Thélèn [7] was the first to notice that p-closed fields are large, see also Jarden [34], Ch. 5, and Pfister, [51] for more on p-closed fields.

Theorem 1.3. *Every p-closed field k is a large field.*

Proof. Let C be a smooth irreducible k-curve with $P \in C(k)$ a k-rational point. Then the normal completion X of C is projective and regular, and $P \in X(k)$ is a smooth point of X. Since $X \backslash C$ is finite, it follows that $C(k)$ is infinite if and only if $X(k)$ is infinite. Thus it is sufficient to show that $X(k)$ is infinite. To do so, recall that by Rosenlicht's Riemann–Roch Theorem, see e.g. Rosenlicht [65], for every $n >> 0$ one has $\dim(nP) = n\deg(P) - \pi_C + 1$, where π_C is Rosenlicht's "genus" of X. Thus since $\deg(P) = 1$, it follows that $\dim(nP) = n - \pi_C + 1$ for every sufficiently large n. Therefore, for each sufficiently large n, there exists f with $(f)_\infty = nP$. On the other hand, for every $a \in k$, the zero divisor $(f-a)_0$ of the function $f - a \in k(X)$ viewed as a rational function on X is of the form $(f-a)_0 = m_1 P_1 + \cdots + m_r P_r$ with $P_1, \ldots, P_r$ closed points of X, and $(f-a)_\infty = nP$. Thus the fundamental equality gives:

$$n = \deg(f-a)_\infty = \deg(f-a)_0 = m_1[\kappa(P_1) : k] + \cdots + m_r[\kappa(P_r) : k].$$

We notice that since $\kappa(P_i)$ are finite extensions of k, and k is a p-closed field, it follows that the degrees $[\kappa(P_i) : k]$ are powers of p. Now choose n sufficiently large and prime to p. Since p does not divide n, it follows that there exists some i such that $[\kappa(P_i) : k]$ is not divisible by p, thus one must have $[\kappa(P_i) : k] = 1$. Equivalently, $\kappa(P_i) = k$ and P_i is a k-rational point of X. Thus we conclude that for every $a \in k$ there exists a k-rational point $P_a \in X(k)$ which is a zero of $f - a \in k(X)$ viewed as a rational function on X. Clearly, for $a \neq b$ from k, the points $P_a, P_b \in X(k)$ are distinct. We conclude that the set $\{P_a\}_{a\in k} \subset X(k)$ is in bijection with k, thus infinite. Thus $C(k)$ is infinite as well. □

D) *Quotient fields of generalized Henselian rings*

Let R be a commutative ring with identity, $\mathfrak{a} \subset R$ be a proper ideal, and setting $\overline{R} := R/\mathfrak{a}$, let $R[t] \to \overline{R}[t]$, $f(t) \mapsto \overline{f}(t)$, be the reduction map for polynomials. Further recall that $a \in R$ is called a simple root of $f(t) \in R[t]$, if $f(a) = 0$ and $f'(a) \in R^\times$. One says that R is Henselian with respect to the ideal $\mathfrak{a}$, or that $(R, \mathfrak{a})$ is a Henselian pair, if the following equivalent conditions hold:

i) For every $f(t)$, every simple root $\overline{a} \in \overline{R}$ of $\overline{f}(t)$ is the specialization of a simple root $a \in R$ of $f(t)$.

ii) For every $f(t)$, if $0 \in \overline{R}$ is a simple root of $\overline{f}(t)$, then $f(t)$ has a simple root in $\mathfrak{a}$.

Recall that if $0, 1 \in \overline{R}$ are the only idempotents in $\overline{R}$, then i), ii) are equivalent to:

(∗) For every $f(t)$, if $0 \in \overline{R}$ is a simple root of $\overline{f}(t)$, then $f(t)$ has a root in the total ring of fractions Quot(R) of R –which is a field if and only if R is a domain.

We notice that if $(R, \mathfrak{a})$ satisfies condition (∗), then $\mathfrak{a}$ is contained in all the maximal ideals of R, thus $1 + \mathfrak{a}$ consists of units in R. See Lafon [42] and Raynaud [63] for the theory of general Henselian rings.

Examples 1.4. The following are examples of generalized Henselian rings:

1) If $R = \varprojlim_n R/\mathfrak{a}^n$ is $\mathfrak{a}$-adically complete, then R is Henselian with respect to $\mathfrak{a}$. In particular:

   a) If $(R, \mathfrak{m})$ is a Noetherian complete local ring, then R is Henselian with respect to $\mathfrak{m}$.

   b) The power series rings $R = R_0[[x_1, \ldots, x_n]]$ are Henselian with respect to $\mathfrak{a} = (x_1, \ldots, x_n)$.

2) Let $S := R_0[x_1, \ldots, x_n] \hookrightarrow R_0[[x_1, \ldots, x_n]] =: R$ be the canonical map, and Quot$(S) \hookrightarrow$ Quot(R) be the resulting embedding of quotient fields. We denote by S^h the relative separable algebraic closure of S in R, i.e., there set of all the elements of R which are separably algebraic over Quot(S). Setting $\mathfrak{a} := (x_1, \ldots, x_n) \subset R$ and $\mathfrak{b}^h := S^h \cap \mathfrak{a}$, it follows directly by the definitions that S^h is Henselian with respect to $\mathfrak{b}^h$.

3) If k is a Henselian field with respect to a non-trivial valuation v, then its valuation ring R_v is Henselian w.r.t. the valuation ideal $\mathfrak{m}_v$.

Remarks 1.5.

1) If $(R, \mathfrak{a})$ is a Henselian pair, then $k = \mathrm{Quot}(R)$ is in general not a Henselian valued field. This happens for instance if R is Noetherian and $\mathrm{Krull.dim}(R) > 1$.

2) It was speculated that the quotient fields of complete local rings, like $k((t_1, t_2)) := \mathrm{Quot}(k[[t_1, t_2]])$, were not large fields. On the other hand, by work of Harbater–Stevenson [32], and more generally by Paran [49], it follows that Problem 4.1 from Part II of this survey has a positive answer over such fields, and that was viewed as new evidence for the solvability of Problem 4.1 over arbitrary base fields k. Unfortunately, the fields $k((t_1, \dots, t_n)) := \mathrm{Quot}\big(k[[t_1, \dots, t_n]]\big)$ as well as those considered by Paran [49] are in fact large fields (thus they do not give new evidence for the solvability of Problem 4.1 in general), by the following:

Theorem 1.6. *Let R be a domain which satisfies condition $(*)$ with respect to some ideal $\mathfrak{a} \neq (0)$. Then the quotient field $k = \mathrm{Quot}(R)$ of R is a large field.*

Proof. Let C be an irreducible smooth k-curve, and $P \in C_{\mathrm{sm}}(k)$. By Fact/Notations I, 2) above, C is birationally equivalent to a plane k-curve $C_0 = V(f) \subset \mathbb{A}^2_k$ such that $f \in k[X_1, X_2]$ is an irreducible polynomial of the form $f(X_1, X_2) = \delta X_2 + \widetilde{f}$, where $\delta \neq 0$ and $\widetilde{f} \in k[X_1, X_2]$ with vanishing terms in degrees < 2. In particular, the origin $P_0 = (0,0)$ is a smooth point of C_0. On the other hand, since $k = \mathrm{Quot}(R)$ is the field of fractions of R, after "clearing denominators," we can suppose that $\delta \in R$ and $f \in R[X_1, X_2]$. Thus the change of variables $X_1 = \delta Y_1$, $X_2 = \delta Y_2$ gives $f(\delta Y_1, \delta Y_2) = \delta^2 h(Y_1, Y_1)$ where $h(Y_1, Y_2) = Y_2 + \widetilde{h}(Y_1, Y_2)]$ for some $\widetilde{h} \in R[Y_1, Y_2]$ having vanishing terms in degrees < 2, and a canonical embedding $C_0 = \mathrm{Spec}\, k[Y_1, Y_2]/(h) \hookrightarrow \mathrm{Spec}\, k[Y_1, Y_2] = \mathbb{A}^2_k$. Thus setting $h_a(t) := h(a, t)$ for $a \in \mathfrak{a}$, we get: $h_a(0) \in \mathfrak{a}$, and $h'_a(0) \in 1 + \mathfrak{a}$, thus $0 \in \overline{R}$ is a simple root of $\overline{h}(t)$. Since R satisfies $(*)$ with respect to $\mathfrak{a}$, there exists a root $b \in k$ of $h_a(t)$, i.e., $h_a(b) = 0$. Equivalently, $h(a, b) = 0$, i.e., (a, b) defines a k-rational point of C_0. Moreover, the set of rational points of this form is in bijection with $\mathfrak{a}$. Thus since $\mathfrak{a} \subset R$ is a non-zero ideal, and R is a domain, one has $|\mathfrak{a}| = |R|$, and $|R| = |k|$. Thus finally $C_0(k)$ has cardinality $|k|$, thus infinite. This concludes the proof. $\square$

Generalizations. Concerning Theorem 1.6 above, one could go on and do the following exercise in generalizing things (although it is not clear whether in this way one gets any new objects of interest). Let R, τ be a commutative topological ring with 1, e.g., τ is an adic topology —in which a basis of neighborhoods of 0 is given by a filtered set of *proper* ideals of R, or a semi-norm topology —thus defined by a semi-norm $\|\ \| : R \to \mathbb{R}$ satisfying $0 = \|0\| \leq \|a\|$, and $\|a + b\| \leq \|a\| + \|b\|$, $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in R$. Given a neighborhood $\mathcal{U}$ of $0 \in R$, the $\mathcal{U}$-neighborhood of $f(t) = \sum_i t^i \in R[t]$ is the set $\mathcal{U}_f$ of all the polynomials $g(t) = \sum_i b_i t^i$ with $\deg(g) \leq \deg(f)$ and $b_i - a_i \in \mathcal{U}$ for all i.

A topological ring R, τ is weakly τ-Henselian, if for every $f(t) \in R[t]$ with $f(0) = 0$ and $f'(0) = 1$ there exists a neighborhood $\mathcal{U}$ of $0 \in R$ such that every $g(t) \in \mathcal{U}_f$ has a zero in the total ring of fractions $\mathrm{Quot}(R)$ of R. And R is called τ-Henselian, if for every $f(t)$ as above, and every neighborhood $\mathcal{U}_0$ of $0 \in R$, there exists a neighborhood $\mathcal{U}$ of $0 \in R$ such that every $g(t) \in \mathcal{U}_f$ has a zero in $\mathcal{U}_0$. And notice that for a given $f(t)$, if $\mathcal{U}_0$ and $\mathcal{U}$ are sufficiently small, then the root of every $g(t) \in \mathcal{U}_f$ in $\mathcal{U}_0$ is unique. In particular, if $(g_\mathcal{U})_\mathcal{U}$ is a generalized sequence in $R[t]$ which converges coefficient-wise to f and has $\deg(g_\mathcal{U}) \leq \deg(f)$, then $g_\mathcal{U}$ has a root $x_\mathcal{U}$ such that the generalized sequence $(x_\mathcal{U})_\mathcal{U}$ tends to $0 \in R$.

Examples 1.7.

1) Let $\mathfrak{a} \subset R$ be an ideal, and τ be the $\mathfrak{a}$-adic topology on R. If $(R, \mathfrak{a})$ is a Henselian pair, then R is τ-Henselian. But R might be (weaky) τ-Henselian in spite of the fact that $(R, \mathfrak{a})$ is not Henselian.

2) Let $R, \|\ \|$ be a semi-normed ring, and τ be the $\|\ \|$-topology on R. If R is $\|\ \|$-Henselian in the sense of Fehm–Paran [15], then R is weakly τ-Henselian, but the converse does not need to hold. And note that a *complete* normed ring R is τ-Henselian, see e.g., Denef–Harbater [9], section 2, especially Theorems 2.1 and 2.3, and Corollary 2.2 (partially reproved in [15] as well).

3) Suppose that R is a (weakly) τ-Henselian ring, and $R_0 \subset R$ is a relatively algebraically closed subring, i.e., $R_0 = \mathrm{Quot}(R_0) \cap R$, and if $a \in \mathrm{Quot}(R)$ is a root of a non-trivial polynomial over R_0, then $a \in \mathrm{Quot}(R_0)$. Then S_0 is (weakly) τ_0-Henselian, where τ_0 is the restriction of τ to R_0.

The main point of this discussion is that for weakly τ-Henselian domains, the proof of Theorem 1.6 works without any changes, see also Fehm–Paran [15] for the case of $\| \ \|$-Henselian rings, and one has:

Theorem 1.8. *Let R be a non-discrete weakly τ-Henselian domain. Then $k := \mathrm{Quot}(R)$ is a large field.*

Remark 1.9. We finally notice that the *non-discrete* τ-Henselianity is –from the logical compactness point of view– the same as the Henselianity with respect to a *non-zero* ideal. In order to explain that, let us first recall terminology and basic facts as follows. Given an extension of rings $R \hookrightarrow \tilde{R}$, we say that R is existentially closed in $\tilde{R}$, notation $R \prec_\exists \tilde{R}$, if every existential sentence with parameters from R which holds in $\tilde{R}$ holds in R as well. Obviously, if $R_1 \subset \tilde{R}$ is a subring such that the image of $R \hookrightarrow \tilde{R}$ lies in R_1, then $R \prec_\exists \tilde{R}$ implies that $R \prec_\exists R_1$ as well. See e.g. Bell–Slomson [5] for definitions, etc.

Now let R be a non-discrete τ-Henselian commutative ring, and $\{\mathcal{U}_i\}_{i\in I}$ be a basis of open neighborhoods of $0 \in R$. Since the intersection $\mathcal{U}_i \cap \mathcal{U}_j$ is non-empty for any i, j, there exist ultrafilters $\mathfrak{D}$ on I which defines convergence to 0, i.e., for every generalized sequence $(x_i)_i$ of elements of R one has: $x_i \to 0$ in the topology τ if and only if for every $\mathcal{U}_j$ one has: $\{i \mid x_i \in \mathcal{U}_j\} \in \mathfrak{D}$. For such an ultrafilter $\mathfrak{D}$, we let ${}^*R = R^I/\mathfrak{D}$ be corresponding ultrapower of R. Let $R_0 := R + \mathfrak{a}_0$ be the subring of ${}^*R$ generated by the image of $R \hookrightarrow {}^*R$ and the set $\mathfrak{a}_0$ of all the ($\mathfrak{D}$-equivalence classes of) generalized sequences $(x_i)_i/\mathfrak{D}$ such that $x_i \to 0$ in the topology τ. Then $\mathfrak{a}_0 \subset R_0$ is in fact an ideal, and the fact that R is τ-Henselian implies directly that R_0 is henselian with respect to $\mathfrak{a}_0$. Since $R \hookrightarrow {}^*R$ is an elementary embedding which factors through $R_0 \hookrightarrow {}^*R$, it follows that $R \prec_\exists R_0$. Hence we conclude the following:

> *For every non-discrete τ-Henselian ring R there exists an existentially closed ring extension $R \prec_\exists R_0$ such that R_0 is Henselian with respect to some ideal $\mathfrak{a}_0 \neq 0$.*

Problem 1.10. *Prove/disprove that $\mathbb{Q}^{\mathrm{ab}}$ is a large field.*

2. Characterizations and Properties of Large Fields

There are several equivalent assertions to the property of being a large field, both of geometrical/arithmetical nature and of model theoretical nature, some of which we describe below. We give complete proofs here, but see Pop [59], Kuhlmann [41], Jarden [34], Ch. 5, for further details and/or somewhat different proofs. Kuhlmann [41] contains a detailed account of the relation between existentially closed field extensions, local uniformization à la Zariski, and large fields.

We begin by noticing that being a large field is a *first order property,* i.e., the property of being a large field can be described using an explicit (countable) scheme of axioms as follows:

Proposition 2.1. *For an arbitrary field k the following are equivalent:*

i) *The field k is a large field.*

ii) *For every irreducible polynomial $f = X_2 + \sum_{i+j>1} a_{ij}X_1^iX_2^j \in k[X_1, X_2]$ and every finite set $S \subseteq k$ there exist $a_1, a_2 \in k$ with $f(a_1, a_2) = 0$ and $a_1 \notin S$.*

In particular, being a large field is an elementary property.

Proof. First, recall that by Fact/Notations I, 2) above, given an irreducible k-curve C and a smooth k-rational point $P \in C(k)$, there exists an affine curve $C_0 \subset \mathbb{A}^2_k$ of the form:

$$C_0 = V(f) \subset \mathbb{A}^2_k, \quad f = X_2 + \textstyle\sum_{i+j>1} a_{ij}X_1^iX_2^j \in k[X_1, X_2] \ \text{ irreducible},$$

and a rational map $\varphi : C \dashrightarrow C_0$ which is defined at P and $\varphi(P) = P_0$ with $P_0 = (0, 0)$. In particular, there exist Zariski open dense subsets $U \subset C$ and $U_0 \subset C_0$ with $P \in U$ and $P_0 \in U_0$ and satisfying: U and U_0 are contained in the smooth locus of C, respectively C_0, and the restriction of φ to U gives rise to a k-isomorphism $\varphi : U \to U_0$. Since the complements of U and U_0 in C, respectively C_0 are finite, and φ defines a bijection $U(k) \to U_0(k)$, all $C(k)$, $U(k)$, $C_0(k)$, $U_0(k)$ are simultaneously finite, or infinite.

Conversely, notice that an irreducible polynomial $f = X_2 + \sum_{i+j>1} a_{ij}X_1^iX_2^j \in k[X_1, X_2]$ is either of the form $f = X_2$ or f is not constant in the variable X_1. Further, every irreducible polynomial f as above defines a k-curve $C_0 := V(f) \subset \mathbb{A}^2_k$ which is (absolutely) irreducible and has $P_0 = (0, 0)$ as a smooth point. This completes the proof of the Proposition 2.1. □

We next explain the relation between existentially closed field extensions and the property of being a large field. We begin by recalling the following well known facts, see e.g., Bell–Slomson [5] for details.

Fact 2.2. Let $k \hookrightarrow \lambda$ be a field subextension of a given field extension $k \hookrightarrow \Lambda$.

1) Then existentially closed field extensions have the following properties:

   a) *Transitivity*: $k \prec_\exists \lambda \prec_\exists \Lambda$ implies $k \prec_\exists \Lambda$.

   b) *Subextensions*: $k \prec_\exists \Lambda$ implies $k \prec_\exists \lambda$.

2) Let $k \hookrightarrow k^I/\mathcal{U}$ be a ultra-power of k which is $|\Lambda|$-saturated. Then $k \prec_\exists \Lambda$ if and only if there exists a field k-embedding $\Lambda \hookrightarrow k^I/\mathcal{U}$.

Examples of existentially closed field extensions $k \hookrightarrow \Lambda$ are the extensions of real closed fields, or of p-adically closed fields satisfying $\Lambda \cap \overline{k} = k$, etc. Important examples of existentially closed field extensions which will be used below are the following: Let $k(t)^{\rm h}$ and $k((t))$ be the Henselization, respectively the completion, of the rational function field $k(t)$ at the t-adic place. Then $k(t)^{\rm h} \prec_\exists k((t))$ under the canonical embedding $k(t)^{\rm h} \hookrightarrow k((t))$, see e.g. Kuhlmann [40] for more about this. More generally, the famous *Artin algebraization conjecture* proved by Popescu [62], but see rather Swan's survey [70], gives a wide generalization of the above assertion, namely: Let R be a local excellent Noetherian domain, and $R^{\rm h} \hookrightarrow \widehat{R}$ be its Henselization, respectively completion. Then $R^{\rm h} \prec_\exists \widehat{R}$, and therefore one has $\mathrm{Quot}(R^{\rm h}) \prec_\exists \mathrm{Quot}(\widehat{R})$ under the embedding defined by $R^{\rm h} \hookrightarrow \widehat{R}$. Note that the local Northerian ring $R := k[t]_{(t)}$ is obviously excellent, and $k(t)^{\rm h} = \mathrm{Quot}(R^{\rm h})$ and $k((t)) = \mathrm{Quot}(\widehat{R})$, thus $k(t)^{\rm h} \prec_\exists k((t))$ is a special case of Popescu's result [62].

Another well known interesting point is the following relation between rational points on k-varieties and existentially closed field extensions:

Fact 2.3. Let X be an irreducible k-variety with function field $k \hookrightarrow k(X)$. Then $k \prec_\exists k(X)$ if and only if the set of k-rational points $X(k)$ of X is Zariski dense in X.

Proof. Set $K := k(X)$ viewed as a field extension of k. First, suppose that $k \prec_\exists k(X) = K$, and let $\eta \in X$ be the generic point of X. Then $\kappa(\eta) = K$, thus η is a K-rational point of X. Further, if $U \subset X$ is any Zariski open k-subvariety, then $\eta \in U$, and in particular, η is a K-rational point of U. Since $k \prec_\exists K$, it follows that U has k-rational points as well. Hence $U(k)$ is non-empty for all open non-empty subsets $U \subset X$, thus $X(k)$ is Zariski dense. Conversely, let $X(k)$ be Zariski dense. Then $U(k)$ is non-empty for every Zariski open non-empty subset $U \subset X$, thus $\big(U(k)\big)_U$ is a pre-filter on $X(k)$. Let $\mathcal{U}$ be any ultrafilter on $X(k)$ containing $\big(U(k)\big)_U$. We claim that there exists a (canonical) k-embedding $k(X) \hookrightarrow k^{X(k)}/\mathcal{U}$. Indeed, for $f \in k(X)$ let $U_f \subset X$ be its domain, and define $\tilde{f} : X(k) \to k$ by $\tilde{f}(x) = f(x)$ if $x \in U_f(k)$, and $\tilde{f}(x) = 0$ if $x \in X(k)\backslash U_f(k)$. Then $k(X) \to k^{X(k)}/\mathcal{U}$, $f \mapsto (\tilde{f}(x))_x/\mathcal{U}$, defines the k-embedding we are looking for. □

We are now in the position to give the following characterization of large fields, see also Kuhlmann [41], where condition ii) below for $d = 1$ was taken to be the definition of a large field:

Proposition 2.4. *For a field k and $d \geq 1$, the following assertions are equivalent:*

i) *k is a large field.*

ii) *$k \prec_\exists k((t_1))\dots((t_d))$ under the canonical embedding $k \hookrightarrow k((t_1))\dots((t_d))$.*

Proof. We first prove the following:

Lemma 2.5. *The field k is large if and only if $k \prec_\exists k(t)^{\rm h}$ under $k \hookrightarrow k(t)^{\rm h}$.*

Proof of Lemma 2.5. For the direct implication "⇒" let $k^I/\mathcal{U}$ an ultra-power of k which is $|k(t)^{\rm h}|$-saturated. By general model theoretic non-sense, it is enough to show that every finitely generated subfield $K \subset k(t)^{\rm h}$ has a k-embedding in $k^I/\mathcal{U}$. Let $K \subset k(t)^{\rm h}$ be such a k-subfield. Then K is a function field in one variable over k, and let C be the unique complete normal –thus regular– k-curve with function field $k(C) = K$. Then the canonical k-place of $k(t)^{\rm h}$ gives rise by restriction to a k-rational place of K, hence to a k-rational point $P \in C(k)$. Since C is regular, P is a smooth point of C. Thus k large implies that $C(k)$ is infinite, hence Zariski dense. Conclude by applying Facts 2.3 and 2.2.

For the converse implication "$\Leftarrow$" let C be a smooth k-curve, $P \in C(k)$ a k-rational point, and t_P a uniformizing parameter at P. Then the function field $k(C)$ is k-embeddable in $k((t_P))$, thus in $k(t_P)^{\mathrm{h}}$ as the latter is the relative algebraic closure of $k(C)$ in $k((t_P))$. Since k is existentially closed in $k(t)^{\mathrm{h}}$ and $k(t_P)^{\mathrm{h}} \cong k(t)^{\mathrm{h}}$ it follows by Fact 2.2 that there exists some k-embedding of $k(t_P)^{\mathrm{h}}$ in some ultra-power of k. That k-embedding gives by restriction a k-embedding of $k(C) \subset k(t_P)^{\mathrm{h}}$ in that ultra-power of k. Applying Fact 2.2 again, we get that $k \prec_\exists k(C)$. Thus $C(k)$ is Zariski dense by loc.cit., hence infinite. This concludes the proof of Lemma 2.5.

Coming back to the proof of Proposition 2.4, one proceeds by induction on d as follows:

For $d = 1$, we have: First, suppose that k is large. Then by Lemma 2.5, it follows that $k \prec_\exists k(t)^{\mathrm{h}}$. Since $k(t)^{\mathrm{h}} \prec_\exists k((t))$ by the remarks above, it follows that $k \prec_\exists k((t))$ by the transitivity of existentially closed field extensions. Conversely, suppose that $k \prec_\exists k((t))$. Then since $k(t)^{\mathrm{h}} \subset k((t))$ is a subfield, it follows that $k \prec_\exists k(t)^{\mathrm{h}}$ by the subextension property.

For the induction step, let us set $k_0 := k$ and $k_i := k_{i-1}((t_i))$ for $1 \le i \le d$. Then we have: First suppose that $k_0 := k$ is large. Then by Section 2), Basic examples 2), it follows that the fields k_i, $1 \le i \le d$, are large. In particular, by the first induction step proved above, it follows that $k_{i-1} \prec_\exists k_i$ for all $1 \le i \le d$. Hence by the transitivity of existentially closed field extensions, one has $k = k_0 \prec_\exists k_d = k((t_1))\dots((t_d))$. Conversely, if $k \prec_\exists k((t_1))\dots((t_d))$, then $k \prec_\exists k((t_1))$, because $k((t_1)) \subseteq k((t_1))\dots((t_d))$. Thus k is large by the case $d = 1$. ☐

Proposition 2.6. *For a field k, the following assertions are equivalent:*

i) *k is a large field.*

ii) *For each irreducible k-variety X one has: $X_{\mathrm{reg}}(k) = X_{\mathrm{sm}}(k)$ is either empty or Zariski dense.*

iii) *For each irreducible k-variety X one has: If $X_{\mathrm{sm}}(k)$ is non-empty then $X(k)$ is Zariski dense.*

Proof. First, recall that X irreducible and $X(k)$ non-empty imply that X is geometrically integral, and X geometrically integral implies that $X_{\mathrm{sm}} \subseteq X_{\mathrm{reg}} \subseteq X$ are Zariski open and dense in X. Coming to the proof of Proposition 2.6, we notice that the implications ii) $\Rightarrow$ i), iii), are obvious. Further, for the implication iii) $\Rightarrow$ ii) we notice that X_{sm} is non-empty (because constains a k-rational point), thus $X_{\mathrm{sm}} \subseteq X$ is open and Zariski dense. Thus $X(k)$ Zariski dense implies that $X_{\mathrm{sm}}(k) = X_{\mathrm{sm}} \cap X(k)$ is Zariski dense as well. Finally, for the implication i) $\Rightarrow$ ii), suppose that $x \in X_{\mathrm{sm}}(k)$ is given, and let $(t_1, \dots, t_d)$ be a system of regular parameters at x. Then the completion of the local ring $\mathcal{O}_{X,x}$ of x is canonically k-isomorphic to $k[[t_1, \dots, t_d]]$, and therefore one has k-embeddings

$$k(X) \hookrightarrow \mathrm{Quot}\big(k[[t_1, \dots, t_d]]\big) \hookrightarrow k((t_1))\dots((t_d)).$$

Since k is large, it follows by Proposition 2.4 that $k \prec_\exists k((t_1))\dots((t_d))$, thus $k \prec_\exists k(X)$ as well, because $k(X) \hookrightarrow k((t_1))\dots((t_d))$. Thus $X(k)$ is Zariski dense in X by Fact 2.3. ☐

The next basic fact about large fields is their permanence under algebraic extensions:

Proposition 2.7. *An algebraic extension of a large field is a large field.*

Proof. There are several proofs of this fact (and all such proofs use some variant of the Weil restriction for algebraic varieties). But maybe the simplest proof is using Proposition 2.4: Let k be a large field, and $l|k$ be an algebraic extension. First suppose that $l|k$ is a finite extension. Then $l(t)^{\mathrm{h}}$ is nothing but the compositum $l(t)^{\mathrm{h}} = l\,k(t)^{\mathrm{h}}$ and similarly for ultra-powers one has that $l^I/\mathcal{U}$ is the compositum $l^I/\mathcal{U} = l\,(k^I/\mathcal{U})$. Hence we deduce that every k-embedding $k(t)^{\mathrm{h}} \hookrightarrow k^I/\mathcal{U}$ gives rise canonically to an l-embedding $l(t)^{\mathrm{h}} \hookrightarrow l^I/\mathcal{U}$. Thus conclude that l is a large field by Fact 2.2 above. Finally let $l|k$ be an arbitrary algebraic extension. Let C be a smooth irreducible curve with an l-rational point P. Then there exists a finite sub-extension $l_0|k$ of $l|k$ such that C and P are defined over l_0, i.e., there exists a smooth irreducible curve C_0 with an l_0-rational point P_0 such that $C = C_0 \times_{l_0} l$ and $P = P_0 \times_{l_0} l$. By the first case discussed above, it follows that l_0 is a large field. Since C_0 has P_0 as a smooth l_0-rational point, it follows that $C_0(l_0)$ is infinite. Hence so is $C_0(l) = C(l)$. ☐

Problem 2.8. *Let $l|k$ be a finite field extension. Prove/disprove that if l is large, so is k.*

3. On the Size and Arithmetical Properties of $X(k)$

In this section we address the question about the cardinality of $X(k)$ and the more subtle question about the arithmetical properties of $X(k)$ in the case k is a large field and X is an integral k-variety with a smooth k-rational point. We will conclude by explaining results about the nature of the group of rational points $A(k)$ for abelian varieties over k. The question about the cardinality of $X(k)$ was raised by Harbater, see [31], discussion before Proposition 3.3, whereas several questions about the nature of $A(k)$ seem to originate from Jarden, see [34], Ch. 5, for much more about this. Theorem 3.1 below seems to be new, although it is a direct generalization of arguments from the proof of [31], Proposition 3.3.

We begin by discussing the cardinality and the arithmetical properties of $X(k)$ over a large field k. Let $x_0 \in X_{\mathrm{reg}}(k)$ be a fixed k-rational point, hence a smooth point of X, and $(t_1,\dots,t_d)$, $(u_1,\dots,u_d)$ be systems of local regular parameters at x_0. Further let $U_0 \subset U \subseteq X$ be open subsets of X on which $t_1,\dots,t_d,u_1,\dots,u_d$ are defined, and $t_1,\dots,t_d$ have no zeros in U_0. In other words, $t_1,\dots,t_d,u_1,\dots,u_d$ are in $\Gamma(U,\mathcal{O}_X)$, and $t_1,\dots,t_d$ are invertible elements in the ring $\Gamma(U_0,\mathcal{O}_X)$.

Theorem 3.1. (Compare with [Ja], Section 5.4.) *In the above notations, the following hold:*

1) *The map $\Psi : U_0(k)\times U(k) \to k^d$ given by $(a,b)\mapsto \big(u_1(b)/t_1(a),\dots,u_d(b)/t_d(a)\big)$ is surjective. Thus in particular, $|X(k)| = |k|$ and the fibers of Ψ have cardinality $|k|$.*

2) *Let $(k_\alpha)_{\alpha\in I}$ be a finite family of proper subfields of k containing a subfield k_0 of cardinality $|k_0| > |I|$ over which X, x_0 are defined. Then $|X(k)\setminus \cup_\alpha X(k_\alpha)| = |k|$.*

Proof. To 1): Without loss of generality, we can replace $U_0 \subset U$ by smaller open subsets. Thus setting $\underline{X} := (X_1,\dots,X_{d+1})$, and $\underline{Y} := (Y_1,\dots,Y_{d+1})$ we can consider U and a further isomorphic copy of U, as being affine open subsets of a k-variety of the form $V(f)$ satisfying the requirements from Fact/Notations I, 2) above, say:

$$U \subset V(f) \subset \mathbb{A}^{d+1}_k \quad \text{and} \quad U \subset V(g) \subset \mathbb{A}^{d+1}_k$$

with $t_i = X_i\big(\mathrm{mod}(f)\big)$, $u_i = Y_i\ \big(\mathrm{mod}(g)\big)$, $1\le i\le d$, where $f\in k[\underline{X}]$ and $g \in k[\underline{Y}]$ are of the form: $f = X_{d+1} + (\text{terms of total degree} > 1)$ and $g = Y_{d+1} + (\text{terms of total degree} > 1)$. Further, $U_0\subset U$ is the open subset consisting of all the points where $f_0 = X_1\dots X_d$ does not vanish and the projection on the first d coordinates $U \to \mathbb{A}^d_k$ is quasi finite (even etale), thus in particular has finite fibers. We view $U\times U$ as a closed k-subvariety of $\mathbb{A}^{d+1}_k\times\mathbb{A}^{d+1}_k = \mathbb{A}^{2d+2}_k$ and for given $\underline{c} = (c_1,\dots,c_d)\in k^d$, consider the k-subvariety $X_{\underline{c}}\subset U\times U$ defined by:

$$X_{\underline{c}} = \operatorname{Spec} k[\underline{X},\underline{Y}]/\big(f,g,Y_1-c_1X_1,\dots,Y_d-c_dX_d\big).$$

Notice that the origin $O := (0,\dots,0)$ of $\mathbb{A}^{2d+2}_k$ lies in $X_{\underline{c}}$, and that O is a smooth point of $X_{\underline{c}}$. Indeed, $X_{\underline{c}}$ is defined by $d+2$ equations in the $2n = 2d+2$ dimensional affine space, and the Jacobian matrix at O of the system of polynomials defining $X_{\underline{c}}$ at O has rank $d+2$, and the image $(\tilde t_1,\dots,\tilde t_d)$ of $(t_1,\dots,t_d)$ under the canonical projection

$$k[\underline{X}]/(f)\to k[\underline{X},\underline{Y}]/(f,g,Y_1-c_1X_1,\dots,Y_d-c_dX_d)$$

is a system of regular local parameters at $O\in X_{\underline{c}}$. Since k is large, it follows that $X_{\underline{c}}(k)$ is Zarski dense, hence there exist points in $X_{\underline{c}}(k)$, say $z\in X_{\underline{c}}(k)$ such that $a_i := \tilde t_i(z)\neq 0$ for $1\le i\le d$. Hence setting $z = (\underline{a},\underline{b})$ with $\underline{a} = (a_1,\dots,a_n)$ and $\underline{b} = (b_1,\dots,b_n)$, one has that:

i) $(\underline{a},\underline{b})\in U_0(k)\times U(k)$

ii) For $i = 1,\dots,d$ one has that $a_i = t_i(\underline{a})$, $b_i = u_i(\underline{b})$, and $c_ia_i = b_i$

Thus we conclude that $\underline{c} := (c_1,\dots,c_d) = (b_1/a_1,\dots,b_d/a_d) = \Psi(\underline{a},\underline{b})$, where Ψ is the map defined in assertion 1) of the Theorem. Since $\underline{c}\in k^d$ was arbitrary, it follows that Ψ is surjective, as claimed. Concerning cardinalities, we reason as follows: By contradiction, suppose that $|X(k)| < |k|$. Then one also has that $|U_0(k)|, |U(k)| < |k|$, and therefore, $|U_0(k)\times U(k)| < |k|$. This contradicts the fact that $\Psi : U_0(k)\times U(k)\to k^d$ is surjective. Concerning the cardinality of the fibers of Ψ, we notice that for every given $\underline{c} = (c_1, \dots, c_d) \in k^d$, the fiber of Ψ at $\underline{c}$ consists of all the points in $X_{\underline{c}}(k) \cap \big(U_0(k) \times U(k)\big)$, which is the same as the k-rational points in the Zariski open subset $X_{\underline{c}} \cap (U_0 \times U)$ of $X_{\underline{c}}$. We conclude by the discussion above.

To 2): Since X and x_0 are defined over k_0, there exist curves $C \subset X$ through x_0 such that C is defined over k_0 and x_0 is a smooth point of C. For such a C one has $X(k_\alpha) \cap C(k) = C(k_\alpha)$ for all $\alpha \in I$. Therefore, it is sufficient to prove that $|C(k) \setminus \cup_\alpha C(k_\alpha)| = |k|$. Thus without loss of generality, we can suppose that X is a k-curve, i.e., $d = \dim(X) = 1$. For every positive integer $N > 0$, we consider the N dimensional variant of the 1-dimensional construction from the proof of assertion 1) above as follows: Let t and u be uniformizing parameters at x_0. Let $\underline{X} := (X_1, X_2)$ and $\underline{Y}_\mu := (Y_{1\mu}, Y_{2\mu})$ for $\mu = 1, \dots, N$ be systems of independent variables such that in the notations from above for $d = 1$ one has:

$$U \subset V(f) \subset \mathbb{A}^2_k, \quad U \subset V\big(g(\underline{Y}_\mu)\big) \subset \mathbb{A}^2_k$$

with $f \in k[X_1, X_2]$ and $g(\underline{Y}_\mu)$ of the form given in Fact/Notations I, 2) above, for the case $d = 1$. Further, $U_0 \subset U$ is the open subset consisting of all the points where $f_0 = X_1$ does not vanish, and U is sufficiently small such that the projection on the first coordinate $U \to \mathbb{A}^1_k$ is quasi finite. We view $U \times U^N$ as a closed k-subvariety of $\mathbb{A}^2_k \times (\mathbb{A}^2_k)^N = \mathbb{A}^{2+2N}_k$ and for every $\underline{c} = (c_1, \dots, c_N) \in k^N$, consider the k-subvariety $U_{\underline{c}} \subset U \times U^N$ defined by:

$$U_{\underline{c}} = \operatorname{Spec} k[\underline{X}, \underline{Y}_1, \dots, \underline{Y}_N] / \big(f, g(\underline{Y}_1), \dots, g(\underline{Y}_N), Y_{11} - c_1 X_1, \dots, Y_{1N} - c_N X_1\big).$$

Notice that the origin $O := (0, \dots, 0)$ of $\mathbb{A}^{2N+2}_k$ lies in $U_{\underline{c}}$, and that O is a smooth point of $U_{\underline{c}}$. Indeed, $U_{\underline{c}}$ is defined by $2N + 1$ equations in the $2N + 2$ dimensional affine space, and the Jacobian matrix at O of the system of polynomials defining $U_{\underline{c}}$ at O has rank $2N + 1$. Further, the image $\tilde{t}$ of t under the canonical projection

$$k[\underline{X}]/(f) \to k[\underline{X}, \underline{Y}_1, \dots \underline{Y}_N] / \big(f, g(\underline{Y}_1), \dots, g(\underline{Y}_N), Y_{11} - c_1 X_1, \dots, Y_{1N} - c_N X_1\big)$$

is a uniformizing parameter at $O \in X_c$. Since k is large, it follows that $U_{\underline{c}}(k)$ is Zariski dense, hence there exist points $z \in U_{\underline{c}}(k) \subset U_0(k) \times U(k)^N$, say $z = (\underline{a}, \underline{b}_1, \dots, \underline{b}_N)$ such that $\underline{a} = (a_1, a_2)$ has $a_1 \neq 0$. Thus setting $\underline{b}_\mu = (b_{1\mu}, b_{2\mu})$ for $\mu = 1, \dots, N$ and using the cardinality assertion 1) of the Theorem, we get the following:

i) $|U_{\underline{c}}(k)| = |k|$

ii) Every $(\underline{a}, \underline{b}_1, \dots, \underline{b}_N) \in U_{\underline{c}}(k)$ satisfies $\underline{c} = (c_1, \dots, c_N) = (b_{11}/a_1, \dots, b_{1N}/a_1)$.

Lemma 3.2. *For every $c \in k \backslash k_0$ and distinct $a_{0\mu} \in k_0$, $1 \leq \mu \leq N$, there exist $b_{11}, \dots, b_{1N}$ in $U(k)$ such that $c = (b_{1\nu} a_{0\mu} - b_{1\mu} a_{0\nu}) / (b_{1\mu} - b_{1\nu})$ for all $1 \leq \mu < \nu \leq N$.*

Proof. Indeed, set $c_\mu = a_{0\mu} + c$. Then every point $(\underline{a}, \underline{b}_1, \dots, \underline{b}_N) \in U_{\underline{c}}$ satisfies the conditions ii) above, thus $a_{0\mu} + c = c_\mu = b_{1\mu}/a_1$ for $\mu = 1, \dots, N$. Therefore, one has that $(a_{0\nu} + c)/(a_{0\mu} + c) = b_{1\nu}/b_{1\mu}$, or equivalently, $c = (b_{1\nu} a_{0\mu} - b_{1\mu} a_{0\nu}) / (b_{1\mu} - b_{1\nu})$ for every $1 \leq \mu < \nu \leq N$. □

Coming back to the proof of assertion 2) of the Theorem, set $N = |I| + 1$. Since $|I| < |k_0|$ by hypothesis, one has $|I| < N \leq |k_0|$. Therefore, we can and will choose N fixed distinct elements $a_{01}, \dots, a_{0N} \in k_0$. For every $c \in k \backslash k_0$, we set $c_\mu = c + a_{0\mu}$ for $\mu = 1, \dots, N$ and notice that $c_1, \dots, c_N \neq 0$, because $c \notin k_0$. We set $\underline{c} := (c_1, \dots, c_N) \in k^N$ and have a closer look at the fibers $U_{\underline{c}}(k)$ of the map hinted at above:

$$\Psi_N : U_0(k) \times U(k)^N \to k^N \quad \text{by} \quad (\underline{a}, \underline{b}_1, \dots, \underline{b}_N) \mapsto (b_{11}/a_1, \dots, b_{1N}/a_1).$$

First, by Lemma 3.2, it follows that for all $1 \leq \mu < \nu \leq N$ one has $c = (b_{1\nu} a_{0\mu} - b_{1\mu} a_{0\nu}) / (b_{1\mu} - b_{1\nu})$. Thus in particular, <u>if</u> there exists some $\mu < \nu$ and some k_α such that $b_{1\mu}, b_{1\nu} \in k_\alpha$, then $c \in k_\alpha$. Second, since the inclusion $k_\alpha \subset k$ is strict for each $\alpha \in I$, viewing $(k, +)$ and $(k_\alpha, +)$ as k_0-vector spaces, it follows that $\cup_\alpha k_\alpha \subset k$ strictly (because $|k_0| > |I|$, thus the k_0-vector space k cannot be the union of its proper k_0-subspaces k_α, $\alpha \in I$). In particular, for $c \in k \setminus \cup_\alpha k_\alpha$, it follows by the previous discussion that for every $\alpha \in I$ there exists at most one μ such that $b_{1\mu} \in k_\alpha$, thus in particular, at most one μ such that $\underline{b}_\mu \in U(k_\alpha)$. Since $N > |I|$, we conclude that for every $(\underline{a}, \underline{b}_1, \dots, \underline{b}_N) \in U_{\underline{c}}(k)$ there must exist some μ such that $b_{1\mu} \notin \cup_\alpha k_\alpha$, and in particular, $\underline{b}_\mu \notin \cup_\alpha U(k_\alpha)$. To fix notations, let $U_{c\mu} \subset U_{\underline{c}}(k)$ be the set of all the points $(\underline{a},\underline{b}_1,\dots,\underline{b}_N)\in U_{\underline{c}}(k)$ such that $\underline{b}_\mu\notin\cup_\alpha U(k_\alpha)$. Notice that by the discussion above, one has $U_{\underline{c}}(k)=\cup_\mu U_{c\mu}$. Next recall that $|U_{\underline{c}}|=|k|$ by assertion 1) of the Theorem, thus there exists some μ such that $|U_{c\mu}|=|k|$. Further, since the projection on the first component $U\to\mathbb{A}^1_k$ was supposed to be quasi finite, hence $U_0\to\mathbb{A}^1_k$ and $U_0\times U^N\to\mathbb{A}^1\times\mathbb{A}^N=\mathbb{A}^{N+1}$ are quasi finite as well, the map

$$\Psi^1: U_0(k)\times U(k)^N\to k^\times\times k^N \quad \text{by} \quad (\underline{a},\underline{b}_1,\dots,\underline{b}_N)\mapsto(a_1,b_{11}\dots,b_{1N})$$

has finite fibers. Since $U_{\underline{c}}\subset U_0\times U^N$ and $U_{c\mu}\subset U_{\underline{c}}(k)$, it follows that $U_{\underline{c}}(k)\to\Psi^1\big(U_{\underline{c}}(k)\big)$ and $U_{c\mu}\to\Psi^1(U_{c\mu})$ have finite fibers as well. Thus since $|U_{c\mu}|=|k|$, one gets $|\Psi^1_N(U_{c\mu})|=|k|$.

Next consider the projection $k^\times\times k^N\to k$, $(a_1,b_{11},\dots,b_{1N})\mapsto b_{1\mu}$. We claim that its restriction $q_\mu:\Psi^1_N(U_{c\mu})\to k$ to $\Psi^1_N(U_{c\mu})$ is injective: Indeed, suppose that $(a_1,b_{11},\dots,b_{1N}),(a'_1,b'_{11},\dots,b'_{1N})$ from $\Psi^1_N(U_{c\mu})$ have equal μ^{th} coordinates $b_{1\mu}=b'_{1\mu}$. Since $(b_{11}/a_1,\dots,b_{1N}/a_1)=\underline{c}=(b'_{11}/a'_1,\dots,b'_{1N}/a'_1)$ and $b_{1\mu}=b'_{1\mu}$, it follows that $a_1=a'_1$. But then one must have $(b_{11},\dots,b_{1N})=(b'_{11},\dots,b'_{1N})$ as well, thus $(a_1,b_{11},\dots,b_{1N})=(a'_1,b'_{11},\dots,b'_{1N})$. Therefore one has that

$$|q_\mu\big(\Psi^1_N(U_{c\mu})\big)|=|\Psi^1_N(U_{c\mu})|=|k|.$$

We next notice that $q_\mu\circ\Psi^1_N:U_0(k)\times U(k)^N\to k$ factors through the μ-projection

$$p_\mu: U_{c\mu}\hookrightarrow U_0(k)\times U(k)^N\to U(k), \quad (\underline{a},\underline{b}_1,\dots,\underline{b}_N)\mapsto\underline{b}_\mu,$$

because $q_\mu\circ\Psi^1(\underline{a},\underline{b}_1,\dots,\underline{b}_N)=b_{1\mu}$ is the first coordinate of $\underline{b}_\mu=p_\mu(\underline{a},\underline{b}_1,\dots,\underline{b}_N)$. Hence since $\Psi^1_N: U_{c\mu}\to\Psi^1_N(U_{c\mu})$ has finite fibers and $q_\mu:\Psi^1_N(U_{c\mu})\to k$ is one-to-one (thus has finite fibers as well), it follows that $p_\mu:U_{c\mu}\to U(k)$ has finite fibers as well. Thus we conclude that $|p_\mu(U_{c\mu})|=|k|$. On the other hand, by the definition of $U_{c\mu}\subset U_{\underline{c}}(k)$ one has that $(\underline{a},\underline{b}_1,\dots,\underline{b}_N)\in U_{c\mu}$ if and only if $\underline{b}_\mu\in U(k)\setminus\cup_\alpha U(k_\alpha)$. Since the last condition is equivalent to $p_\mu(\underline{a},\underline{b}_1,\dots,\underline{b}_N)\in U(k)\setminus\cup_\alpha U(k_\alpha)$, we conclude that $p_\mu(U_{c\mu})\subset U(k)\setminus\cup_\alpha U(k_\alpha)$. Therefore we have $|k|=|p_\mu(U_{c\mu}|\le|U(k)\setminus\cup_\alpha U(k_\alpha)|\le|k|$, thus $|U(k)\setminus\cup_\alpha U(k_\alpha)|=|k|$, as claimed. This concludes the proof of Theorem 3.1. □

We next discuss the rank of the group of rational points of an abelian variety over a large field. First, let A be an abelian variety of dimension d over some arbitrary base field k, and $\overline{k}$ be an algebraic closure of k. Then $A(\overline{k})$ is an abelian group, whose structure is well known:

a) $A(\overline{k})_{\rm tors}=\big(\mathbb{Q}/\mathbb{Z}\big)^{2d}$ if $\mathrm{char}(k)=0$, and $A(\overline{k})_{\rm tors}=\big(\mathbb{Q}/\mathbb{Z}\big)^{2d}/\big(\mathbb{Q}_p/\mathbb{Z}_p\big)^{2d-r_A}$ if $\mathrm{char}(k)=p>0$, where r_A is the Hasse–Witt invariant of A.

b) $A(\overline{k})=A(\overline{k})_{\rm tors}$ if and only if k is algebraic over a finite field, and if k is not algebraic over a finite field, then $A(\overline{k})/A(\overline{k})_{\rm tors}$ is a $\mathbb{Q}$ vector space of dimension $|k|=|\overline{k}|$.

In particular, the canonical embedding $A(k)\hookrightarrow A(\overline{k})$ gives hints about the structure of $A(k)$, e.g., about the nature of the torsion subgroup $A(k)_{\rm tors}$, but in general very little is known about the structure of $A(k)$. Recall that given an abelian group G, the rational rank of G is the dimension of the $\mathbb{Q}$-space $G\otimes_{\mathbb{Z}}\mathbb{Q}$. In particular, by the discussion above, it follows that the rational rank of the abelian group $A(\overline{k})$ is either zero, of infinite and equal to $|\overline{k}|$.

Concerning the rational rank of $A(k)$ in general little is known. The famous Mordell–Weil theorem asserts that $A(k)$ is finitely generated for k a number field, whereas in the case k is a function field over some subfield k_0 and the k_0-trace of A is trivial, the same holds for $A(k)$. In particular, in the cases above, $A(k)$ has finite rational rank as well.

Finally, over large fields one has the following.

Theorem 3.3. *Let k be a large field with* $\mathrm{char}(k)=0$*, and A be a positive dimensional abelian variety over k. Then the abelian group $A(k)$ has rational rank equal to $|k|$.*

A first and by far most difficult step in the proof is to prove that $A(k)$ has infinite rank. That was accomplished by Fehm–Petersen [16], by employing high end technical tools, like the (generalized) Mordell–Lang conjecture (see Raynaud [64], Faltings [13], [14], Hindry [33]...). The case where k has infinite transcendence degree was done in Fehm–Petersen [17], and follows from (weaker variants of) Theorem 3.1 above. See [34], Ch. 6 for a detailed exposition of these facts.

Questions/Problems 3.4.

1) *Prove/disprove that the hypothesis $|k_0| > |I|$ is not necessary in Theorem 3.1.*
2) *Give a generalization of Theorem 3.1 to the case of families $(k_\alpha)_{\alpha\in I}$ with I infinite.*
3) *Does Theorem 3.3 hold for all large fields k which are not algebraic over finite fields?*
4) *Give a more elementary proof of Theorem 3.3, which might work in all characteristics.*

PART II: **Applications**

One should mention here that the large fields were introduced and used initially in the context of (inverse) Galois theory in Pop [59], but shortly after one realized that one can do lots of interesting mathematics over large fields. Below is a summary of (some of) these applications of large fields.

4. (Inverse) Galois Theory over Large Fields

Let G be an arbitrary profinite group. Recall that a finite embedding problem EP for G is a diagram of surjective morphisms of profinite groups of the form

$$\mathrm{EP} = (\gamma : G \to A, \alpha : B \to A), \quad B \text{ finite.}$$

We say that EP is split, if $\alpha : B \to A$ has a group theoretical section, and we say that a group homomorphism $\beta : G \to B$ is a (proper) solution of EP, if (β is surjective and) $\gamma = \alpha \circ \beta$. Obviously, if EP is split, then EP has solutions (which are not necessarily proper).

For a base field k, let $k(t)$ be the rational function field in the variable t over k, and G_k and $G_{k(t)}$ be the absolute Galois group of k, respectively $k(t)$. The embedding $k \hookrightarrow k(t)$ gives rise to the canonical projection $pr_t : G_{k(t)} \to G_k$ of absolute Galois groups. Therefore, every finite (split) embedding problem $\mathrm{EP} = (\gamma : G_k \to A, \alpha : B \to A)$ for G_k gives rise canonically to a finite (split) embedding problem EP_t for $G_{k(t)}$ of the form:

$$\mathrm{EP}_t := (\gamma \circ pr_t : G_{K(t)} \to A, \quad \alpha : B \to A).$$

A proper solution β_t of EP_t is called a regular proper solution of EP, if denoting by $k_A \subset k^{\mathrm{sep}}$ the Galois extension of k with $\mathrm{Gal}(k_A|k) = A$, and letting $k(t)_\beta \subset k(t)^{\mathrm{sep}}$ be the Galois extension of $k(t)$ with $\mathrm{Gal}\big(k(t)_\beta|k(t)\big) = B$, it follows that $k(t)_\beta \cap k^{\mathrm{sep}} = k_A$.

The following is arguably the main open problem in Galois theory:

Problem 4.1. *Let k be an arbitrary field. Prove/disprove that for every finite split embedding problem* EP *for G_k, the corresponding* EP_t *for $G_{k(t)}$ has proper solutions.*

The Problem 4.1 above seems to have been a kind of math folklore since the beginning of the 1990's but it was first explicitly stated in Dèbes–Deschamps [8]. Positive answers to the above Problem would imply —among other things, positive answers to the Inverse Galois Problem and the Shafarevich Conjecture on the profinite freeness of the kernel of the cyclotomic character of global fields. Recall that the geometric case of the Shafarevich Conjecture is true by Harbater [29] and Pop [56], "officially" published in [57].

Theorem 4.2. *Let k be a large field. Then every finite split* EP *for G_k has proper regular solutions, and if* $\ker(\alpha)$ *is non-trivial, there are $|k|$ distinct proper regular solutions.*

The result was proved in Pop [59], Main Theorems A, where the large fields were introduced, and further applications of the above Theorem were given, from which we mention: The proof of a conjecture by Roquete asserting that a countable PAC field is Hilbertian if and only if its absolute Galois group is profinite free. (The "only if" implication was known already for quite some time.) The description of the Galois group of the field $k^{\mathfrak{S}}$ of totally $\mathfrak{S}$-adic numbers, where k is a global field, and $\mathfrak{S}$ is a finite set of places of k. Finally, the proof of the semi-local version of the Shafarevich conjecture on the profinite freeness of the kernel of the cyclotomic character of global fields. (There is the hope that that result reduces the Shafarevich conjecture under discussion to a group theoretical conjecture, but there is not much progress on that problem yet.)

Concerning the result above about the existence of proper regular solutions for embedding problems EP for G_k, with the same methods (rigid/formal patching) one can prove stronger assertions, mentioned already in the "officially" unpublished paper Pop [56] (which predated the large fields days). Namely let $K = k(X)$ be the function of a complete normal curve X over k. For a fixed finite (split) embedding problem $\mathrm{EP}_K := (\gamma : G_K \to A, \alpha : B \to A)$ for G_K we denote by $K_A \subset K^{\mathrm{sep}}$ the Galois extension of K with $\mathrm{Gal}(K_A|K) = A$, thus K_A is the fixed field in K^{sep} of $\ker(\gamma)$, and for a proper solution $\beta : G_K \to B$, we let $K_\beta \subset K^{\mathrm{sep}}$ be the finite Galois extension of K with $\mathrm{Gal}(K_\beta|K) = B$. We will say that β is a regular solution of EP_K if $K_A \cap \overline{k} = K_\beta \cap \overline{k}$. The stronger result hinted at above is the following:

- *For k large and $K = k(X)$ as above, every finite split EP_K has proper regular solutions.*

Nevertheless, the subsequent developments showed that over large fields k even stronger assertions hold as we will explain below. First, recall the famous *Dedeking–Weber correspondence* which asserts that the category of complete normal k-curves X and non-constant k-morphisms is (anti)equivalent to the category of function fields $K|k$ with $\mathrm{tr.deg}(K|k) = 1$ and k-embeddings. If X is a complete normal k-curve with function field $K = k(X)$, the following hold:

a) The local rings $\mathcal{O}_{X,x}$ of the points $x \in X$ are precisely the valuation rings $\mathcal{O}_v$ of the (equivalence classes of) k-valuations v of K. If $\mathcal{O}_{X,x} = \mathcal{O}_v$ we say that x and v correspond to each other.

b) The finite k-embeddings $K'|K$ are functorially in bijection with the finite covers $X' \to X$ of complete normal curves. And if v corresponds to $x \in X$, then the prolongations $v'|v$ of v to K' are in bijection with the preimages $x' \mapsto x$ of $x \in X$ under $X' \to X$.

Now suppose that $K'|K$ is a finite Galois extension. We denote $T_{v'|v} \subset Z_{v'|v} \subset \mathrm{Gal}(K'|K)$ the inertia/decomposition groups of $v'|v$ (or the corresponding $x' \mapsto x$). Recall that $T_{v'|v}$ is trivial for all but finitely many $x \in X$, and if $e(v'|v) := (v'K' : vK)$ is the ramification index and $f(v'|v) := [\kappa(v') : \kappa(v)]$ is the residual degree of $v'|v$, then $|Z_{v'|v}| = e(v'|v)f(v'|v)$. Further, $\kappa(v')|\kappa(v)$ is a finite normal extension, and if $f_{\mathrm{i}}(v'|v)$ is the degree of the maximal purely inseparable subextension of $\kappa(v')|\kappa(v)$, one has that $|T_{v'|v}| = e(v'|v)f_{\mathrm{i}}(v'|v)$. Finally, recalling the exact sequence $1 \to T_{v'|v} \to Z_{v'|v} \to \mathrm{Aut}\big(\kappa(v')|\kappa(v)\big) \to 1$, one has: If $T_{v'|v} = 1$, i.e., $v'|v$ is unramified, then $\kappa(v')|\kappa(v)$ is a Galois extension of $\kappa(v)$ with Galois group canonically isomorphic to $Z_{v'|v}$. Furthermore, if $\kappa(v) = k$, or equivalently, the point $x \in X$ corresponding to v is k-rational, then $\kappa(v')|k$ is a Galois extension with Galois group $Z_{v'|v} \subset \mathrm{Gal}(K'|K)$.

Let $\mathrm{EP}_K := (\gamma : G_K \to A, \alpha : B \to A)$ be a fixed finite (split) embedding problem for G_K, and provided a proper solution β of EP_K is given, let $K \hookrightarrow K_A \hookrightarrow K_\beta$ be as defined/introduced above. Let $X_\beta \to X_A \to X$ be the corresponding covers of complete normal curves, and $x_\beta \mapsto x_A \mapsto x$ denote preimages of $x \in X$ in X_β, respectively X_A. Correspondingly, if v is the k-valuation of K corresponding to $x \in X$, let $v_\beta \mapsto v_A \mapsto v$ denote the corresponding prolongations of v to K_β, respectively K_A. Then setting $C := \ker(\alpha)$, the map $\alpha : B \to A$ gives rise to a commutative diagram:

$$\begin{array}{ccccccc} 1 \to & Z_{v_\beta|v_A} & \longrightarrow & Z_{v_\beta|v} & \longrightarrow & Z_{v_A|v} & \to 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \to & C & \longrightarrow & B & \xrightarrow{\alpha} & A & \to 1 \end{array}$$

The refinements of the existence of proper solutions for finite split embedding problems EP_K as introduced above ask for the existence of solutions β which satisfy extra *local conditions* at a given finite set of closed points of X, that is, a finite set Σ of k-valuations of K. Precisely, let exact sub-sequences

$$\mathcal{S}_\Sigma : \quad 1 \to C_v \to B_v \to A_v \to 1, \quad v \in \Sigma,$$

of $1 \to C \to B \to A \to 1$ be given such that $A_v = Z_{v_A|v}$ and $B_v \to A_v$ are quotients of the absolute Galois group G_{K_v} of the completion of K at v. We say that the finite split embedding problem EP_K with local conditions $\mathcal{S}_\Sigma$ is solvable, if there exists a proper solution $\beta : G_K \to B$ such that for every $v \in \Sigma$ the exact sequence $1 \to Z_{v_\beta|v_A} \to Z_{v_\beta|v} \to Z_{v_A|v} \to 1$ from the above diagram is conjugated to the exact sequence $1 \to C_v \to B_v \to A_v \to 1$.

• From now on suppose that k *is a large field.*

The first instance where finite split embedding problems with local conditions were considered seems to be Colliot-Thélène [7]. He considered the case $X = \mathbb{P}^1_k$ with k having $\mathrm{char}(k) = 0$, and showed the following: Let $x \in \mathbb{P}^1(k)$ be a k-rational point which is not branched in $K_A|K$, thus the corresponding valuation v has $\kappa(v) = k$ and $T_{v_A|v} = 1$. Let $1 \to C_v \to B_v \to A_v \to 1$ at v be such that $A_v = Z_{v_A|v}$ and $B_v \to A_v = Z_{v_A|v}$ is an isomorphism. Then there exist proper solutions β such that $v_\beta|v$ is not ramified, and $Z_{v_\beta|v} \to Z_{v_A|v}$ is conjugated to $B_v \to Z_{v_A|v}$. Equivalently, one can prescribe the conjugacy class of the decomposition group $Z_{v_\beta|v}$ of v_β with the only restriction that that conjugacy class maps to the conjugacy class of $Z_{v_A|v}$ and $Z_{v_\beta|v} \to Z_{v_A|v}$ is an isomorphism. And **importantly**, Colliot-Thélène [7] shows that in general there *do not exist* proper solutions β which satisfy local conditions at *more than one point.* Subsequently, it was shown by several people independently, e.g., Moret-Bailly [47], Harbater, Pop (unpublished notes), that the main result from [7] concerning solvability of finite split embedding problems with local conditions at one *single* k-rational point holds in the same form over arbitrary complete geometrically integral normal X over any large field k. Finally, some instances of this aspect of solving finite split embedding problems were (re)proved by Haran–Jarden [23], [24], [25] using "algebraic patching" methods. Before moving on let me summarize the results explained above in a formal way:

Theorem 4.3. *Let k be a large field, and $K = k(X)$ be the function field of a complete geometrically integral normal k-curve. Let $\mathrm{EP}_K = (\gamma : G_K \to A, \alpha : B \to A)$ be a finite split embedding problem for G_K, and v be a k-valuation of K which is not ramified in $K_A|K$ and has $\kappa(v) = k$. Finally, let $B_v \subset B$ be a subgroup which α maps isomorphically onto $Z_{v_A|v}$. Then EP_K has proper regular solutions β for which $Z_{v_\beta|v}$ is conjugated to B_v. Moreover, if $\ker(\alpha)$ is non-trivial, then there exist $|k|$ distinct such solutions.*

Remarks 4.4. We notice that all the above results satisfy quite mild local conditions only, namely the local groups $B_v \to A_v = Z_{v_A|v}$ are always unramified, and $B_v \to A_v$ is an isomorphism. It seems that the strongest result so far in this direction is by Harbater–Pop, see [30], Theorem 5.2.3, where a special case of the situation in which $v_\beta|v$ is unramified, but $B_v \to A_v = Z_{v_A|v}$ is not necessarily injective, was studied. (Because the result is quite technical, we will not go into further details here.)

Questions/Problems 4.5. *See also the list of problems from [2].*

1) *Is it true that k is large if and only if the function fields K of all complete regular k-curves satisfy: Every finite split EP_K for G_K has proper regular solutions?*

2) *Generalize the results above about solvability of EP_K with local conditions by weakening the hypotheses on v as follows:*

   a) *v is unramified in $K_A|K$, $B_v \to Z_{v_A|v}$ is an isomorphism, but $\kappa(v) \neq k$.*
   b) *$\kappa(v) = k$, $B_v \to Z_{v_A|v}$ is an isomorphism, but $T_{v_A|v} \neq 1$.*
   c) *$B_v \to Z_{v_A|v}$ is an isomorphism, but $\kappa(v) \neq k$ and $T_{v_A|v} \neq 1$.*

3) *What about the solvability of EP_K in the case $B_v \to A_v$ is not an isomorphism?*

4) *What about the solvability of EP_K with local conditions at more than one point?*

5. Elementary theory of function fields over large fields

For an arbitrary field K, let $\mathfrak{Th}(K)$ be the elementary (or first order) theory of K, i.e., the set of all the first order sentences in the language of fields which are true in K. Clearly, if two fields are isomorphic as fields $K \cong L$, then $\mathfrak{Th}(K) = \mathfrak{Th}(L)$. The converse of this does not hold, because K is elementarily equivalent to all its ultrapowers ${}^*K := K^I/\mathcal{U}$, thus $\mathfrak{Th}(K) = \mathfrak{Th}({}^*K)$, but K and ${}^*K$ are usually not isomorphic. A basic fact in model theory asserts that $\mathfrak{Th}(K) = \mathfrak{Th}(L)$ if and only if there exist ultrapowers ${}^*K := K^I/\mathcal{U}_I$ and ${}^*L := L^J/\mathcal{U}_J$ which are isomorphic as fields.

Coming back to arithmetic and algebraic geometry, it is one of the most fundamental open questions in the elementary (or first order) theory of *finitely generated fields,* whether $\mathfrak{Th}(K)$ characterizes such a field up to isomorphism, and that question is called the *Elementary equivalence versus isomorphism problem* (EEvIP). Concerning this, one should mention the result by Rumely [66], which asserts that for every *global field* K there exists a sentence φ_K in the language of fields such that for all global fields L one has: φ_K holds in L if and only if $K \cong L$ as fields. See also Pop [60] for more about the case of arbitrary finitely generated fields, Clark [6] for function fields of genus one curves over global fields, and Poonen [54] for further substantial advancement on the subject.

Since every global field is by definition a finitely generated field over its prime field, thus a function field over a prime field, the following question/problem arises: Is it true that the elementary theory $\mathfrak{Th}(K)$ of function fields $K|k$ with $\text{tr.deg}(K|k) > 0$ over "reasonable base fields" k encodes the isomorphism type of $K|k$? The notion of a "reasonable base field" is not formally defined, but the class of such fields should be closed under finite extension and should contain the *prime fields* $\mathbb{F}_p$, $\mathbb{Q}$, and at the other extreme, the *large fields* k. First evidence that the algebraically closed fields might be reasonable base fields are results by Duret [10], [11], Pierce [53], Vidaux [72], [73] (for function fields of curves. See Pop [60], where among other things the case of arbitrary functions fields over an algebraically closed field k was considered. Finally, evidence that some classes of large fields, e.g., the p-adically closed fields, might be reasonable can be found in Bélair–Duret [3], [4], Koenigsmann [35].

The following results give some stronger evidence for the possibility that the large fields (maybe satisfying some cohomological dimension restrictions) should be reasonable base fields in the sense above, see Poonen-Pop [55] for more details about this.

Theorem 5.1.

1) *There exists a formula with one parameter* $\Psi(\mathfrak{x})$ *such that for all large fields* k *and all function fields* $K|k$ *the following holds:* $k' := \{x \in K \mid \Psi(x) \text{ holds in } K\}$ *is the constant field of* K*, i.e., the relative algebraic closure of* k *in* K*.*

2) *For each of the following classes of fields, there is a sentence in the language of fields that is true for fields in that class and false for fields in the other five classes:*

   - *finite or large fields*
   - *number fields*
   - *function fields over finite fields*
   - *function fields over large fields of positive characteristic*
   - *function fields over large fields of characteristic zero*
   - *function fields over number fields*

Remark 5.2. It is impossible to distinguish finite fields from large fields with a single sentence, because a nontrivial ultraproduct of finite fields is a large field.

Fact 5.3. Recall that the Kronecker dimension of a field E is $\text{Kr.dim}(E) := \text{tr.deg}(E|E_0) + \text{Kr.dim}(E_0)$, where E_0 is the prime field of E, and $\text{Kr.dim}(\mathbb{F}_p) := 0$, $\text{Kr.dim}(\mathbb{Q}) := 1$. One has the following:

1) Let E be a field of characteristic $\neq 2$, and set $E' := E[\sqrt{-1}]$. In particular, $\mu_4 = \{\pm 1\}$ is contained in E', thus it is a trivial G_E-module. Let $\text{vcd}_2^0(E)$ be the supremum over all n such that $\text{H}^n(E', \mu_2) \neq 0$. Since the virtual 2-cohomological dimension $\text{vcd}_2(E)$ is defined similarly, but the supremum is taken over all possible 2-torsion $G_{E'}$-modules, and $\text{vcd}_2(E)$ is bounded $\text{Kr.dim}(E)+1$, one has:
$$\text{vcd}_2^0(E) \leq \text{vcd}_2(E) \leq \text{Kr.dim}(E) + 1.$$

2) The Milnor Conjecture (proved by Voevodsky et al) asserts that the n^{th} cohomological invariant $e_n\colon I^n(E)/I^{n+1}(E) \to \text{H}^n(E, \mu_2)$, which maps each n-fold Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$ to the cup product $(-a_1) \cup \cdots \cup (-a_n)$, is a well defined isomorphism. Using the Milnor Conjecture one can describe $\text{vcd}_2^0(E)$ via the behavior of Pfister forms as follows: $\text{vcd}_2^0(E) < n$ if and only if every n-fold Pfister form over E represents 0 over E'. See Pfister's nice survey [52] for more about this.

3) There exist fields E with $\mathrm{vcd}_2^0(E) < \mathrm{vcd}_2(E)$. For instance, let E be a maximal pro-2 Galois extension of a global or local field of characteristic $\neq 2$ containing $\sqrt{-1}$. Then every element of E is a square, so $\mathrm{vcd}_2^0(E) = 0$. On the other hand, since the Sylow 2-groups of G_E are non-trivial, one has $\mathrm{vcd}_2(E) > 0$.

Definition 5.4. A field E is said to be 2-cohomologically well behaved if $\mathrm{char}(E) \neq 2$ and for every finite extension $F|E$ one has that $\mathrm{vcd}_2^0(F) = \mathrm{vcd}_2(F)$ and $\mathrm{vcd}_2(F)$ is finite.

Remark 5.5. In the above notations and context, the following hold:

1) The property of being 2-cohomologically well behaved is closed under taking finitely generated field extensions. That means, if E is a 2-cohomologically well behaved field, then every finitely generated field extension $F = E(x_1, \dots, x_n)$ of E is 2-cohomologically well behaved as well. Further, the finite fields, the global fields and the local fields are 2-cohomologically well behaved. Thus all function fields over such fields are 2-cohomologically well behaved as well.

2) The following assertions are equivalent:

   i) $\mathrm{vcd}_2^0(E) = n$.

   ii) The Pfister n-fold Pfister forms $q_n = \langle\langle a_1, \dots, a_n \rangle\rangle$ over $E' := E[\sqrt{-1}]$ satisfy:

      a) There exist n-fold Pfister forms q_n which do not represent 0.

      b) Every n-fold Pfister form q_n is universal, i.e., it represents all $a \in E'^{\times}$.

3) We notice that the assertions a), b) above are actually sentences in the language of fields. Let φ_n be the (logical) conjunction of the two sentences, thus φ_n is a sentence in the language of fields. Then the fact that $\mathrm{vcd}_2^0(E) = n$ is equivalent to the fact that the sentence φ_n holds in E.

We conclude this section by mentioning the following results concerning characterizing algebraic dependence. Unfortunately, the formulas which characterize the algebraic independence *depend* on the function field $K|k$, thus they are not universal. Nevertheless, the result below could be viewed as a first step towards showing that function fields over large fields which are 2-cohomologically well behaved are "plausible base fields."

Theorem 5.6. *Let k be a large field which is 2-cohomologically well behaved, and $K|k$ be a function field over k. Let $\Psi(\mathfrak{x})$ be the formula introduced in Theorem 5.1, φ_n be the sentences defined above for $n \geq 0$, and $k' \subset K$ be the field of constants defined by the predicate $\Psi(\mathfrak{x})$. Then the following hold:*

1) *The sentence φ_e holds over k' if and only if $\mathrm{vcd}_2^0(E) = e$, and $\mathrm{tr.deg}(K|k) = d$ if and only if there exists some e' such that $\varphi_{e'}$ holds over k' and $\varphi_{e'+d}$ holds over K.*

2) *For every r there exists a first order formula $\Psi_{K|k,r}(\mathfrak{t}_1, \dots, \mathfrak{t}_r)$ depending on K and k with r free variables, in the language of fields, such that for elements $t_1, \dots, t_r \in K$ one has: $\Psi_{K|k,r}(t_1, \dots, t_r)$ holds in K if and only if $(t_1, \dots, t_r)$ are algebraically independent over k.*

The main obstacle in finding absolute formulas which describe the algebraic independence in function fields $K|k$ over large fields k originates from the fact that the length of the formulas $\Psi_{K|k,d}(\mathfrak{t}_1, \dots, \mathfrak{t}_d)$ uses implicitly the birational invariant $\delta_{K|k}$ of $K|k$ obtained as follows: Let X be some model of $K|k$, and for every $n > 0$, let $\Sigma_n \subseteq X$ be the set of closed points $x \in X$ such that 2^n does not divide $[\kappa(x) : k]$. Then for n sufficiently large the set Σ_n is dense in X. We let $\delta_{K|k}$ be the minimum of all n such that Σ_n is Zariski dense in X, and notice that by general scheme theoretical non-sense it follows that $\delta_{K|k}$ is indeed a birational invariant of $K|k$, i.e., does not depend on the model X. On the other hand, when $K|k$ varies, there are in general no upper bounds for the numbers $\delta_{K|k}$, thus there is no bound on the length of the formulas $\Psi_{K|k,r}$ given in Poonen-Pop [55] (because they explicitly depend on $\delta_{K|k}$, thus implicitly on $K|k$). See loc.cit., Propositions 5.5, 5.6 and their proofs for more details.

Questions/Problems 5.7. *In the context of Theorem 5.6 do the following:*

1) *Find absolute formulas, i.e., formulas which do not depend on $K|k$, which describe the algebraic independence over k, provided k is a large field.*

2 *Find formulas which describe the k-valuations of $K|k$ (which have geometrical meaning, e.g., the prime divisors of $K|k$), provided k is a large field.*

6. Characterizing extremal fields

The extremal fields form a special class of valued fields which were introduced by Ershov [12]. But shortly after it was realized that one has to amend the definition (because otherwise the only extremal fields would be the algebraically closed ones). It is quite interesting and surprising that the large fields show up in the characterization of a class of extremal fields.

Definition 6.1. Let K, v be a valued field with valuation ring $\mathcal{O}_v$, and the valuation v written additively. Then K, v is called extremal if for all positive integers $n \geq 1$ and every polynomial $p(X_1, \dots, X_n)$ over $\mathcal{O}_v$, the set of values $v\big(p(\mathcal{O}_v^n)\big) := \{v\big(p(a_1, \dots, a_n)\big) \mid a_1, \dots a_n \in \mathcal{O}_v\}$ has a maximal element in vK.

Remark 6.2.

1) The original definition from [12] asks for a maximal element of the set of all the values $v\big(p(K^n)\big)$ instead of $v\big(p(\mathcal{O}_v^n)\big)$. That condition is not satisfied in many situations of interest, e.g., for K the Laurent power series $K = \mathbb{R}((t))$ endowed with the canonical t-adic valuation and the polynomial $p(X_1, X_2) = X_1^2 + X_1^2 X_2^2 - 2X_1X_2$. In particular, Proposition 2 of [12] does not hold for the original definition from [12].

2) Being an extremal field can be written with the countable scheme of axioms, axiomatizing that *for every $n \geq 1$, and every $p(X_1, \dots, X_n) \in \mathcal{O}_v[X_1, \dots, X_n]$, the set $v\big(p(a_1), \dots, a_n)\big)$, $a_1, \dots, a_n \in \mathcal{O}_v$, has a maximal element.* Therefore, the class of extremal fields is an elementary class.

The seminal paper Azgin–Kuhlmann–Pop [1] proves a few basic properties of the extremal fields, which we summarize below. First, let us recall the following definitions: Let K, v be a valued field. For a finite field extension $L|K$ and a prolongation w of v to L, we let $e(w|v) := (wL : vK)$ and $f(w|v) := [Lw : Kv]$ be the ramification index, respectively the residual degree of $w|v$. Further, the value group vK is called a $\mathbb{Z}$-group, if vK has a minimal positive element $1_v \in vK$ such that $vK/\langle 1_v \rangle$ is a divisible group, where $\langle 1_v \rangle$ is the subgroup of vK generated by 1_v. (Note that $\langle 1_v \rangle$ is the minimal non-trivial convex subgroup of vK.)

Theorem 6.3. *The extremal fields K, v have the following properties:*

1) *K, v is algebraically complete, i.e., v is a Heselian valuation of K, and the fundamental equality $[L : K] = e(L|K)\, f(L|K)$ holds for every finite extension $L|K$ of K.*

2) *The value group vK is either divisible or a $\mathbb{Z}$-group.*

3) *The class of extremal fields is closed under finite extensions.*

The classification problem for extremal fields is very exciting. The assertion 2) of the theorem above produces a first dichotomy in the class of extremal fields via their value group, which is either a $\mathbb{Z}$-group, or a divisible group. The other dichotomy is given by the residue characteristic, namely whether it is zero or positive. The residue characteristic zero case is completely understood, as follows:

Theorem 6.4. *Extremal valued fields with residue characteristic zero are precisely the* henselian valued fields K, v *which satisfy one of the following:*

a) *The value group vK is a $\mathbb{Z}$-group.*

b) *The value group vK is divisible and the residue field Kv is a large field.*

The extremal fields having positive residue characteristic and a $\mathbb{Z}$-group as a value group are quite out of reach for the time being (although examples are known, e.g., the p-adically closed fields), whereas extremal fields having divisible valued group are better understood:

Theorem 6.5. *Let K, v be a valued field with divisible value group vK. Then one has:*

1) *If K, v is extremal, then the residue field Kv of v is a large field.*

2) *Suppose that K has positive characteristic, Kv is large and perfect, and K, v is algebraically complete. Then K is perfect and K, v is extremal.*

It would be very desirable to have a complete characterization of the extremal fields, because the problem is interesting in itself, but also because this class of fields might play an important role in tackling one of the foremost fundamental problems in the model theory of valued fields, namely clarifying the elementary theory of $\mathbb{F}((t))$, where $\mathbb{F}$ is a finite field.

Questions/Problems 6.6. *In the context above, consider the following:*

1) *Give a characterization of the extremal fields of positive characteristic which are not necessarily perfect, but have a divisible value group.*

2) *Same question for extremal fields of characteristic zero having residue field of positive characteristic.*

3) *Give characterizations of the extremal fields whose value groups are $\mathbb{Z}$-groups.*

7. Rationally connected / unirational varieties

Rationally connected varieties are algebraic varieties on which the lines are "abundant" in the following sense: Let k be an algebraically closed field with $\mathrm{char}(k) = 0$. One says that an irreducible k-variety X is rationally connected, if any two "general" closed points of X can be connected by a line which lies inside X. More formally, this means that there exists a Zariski open (affine) subset $U \subset X$ such that for any two closed points $x, y \in U$ there exists a line $\mathfrak{l} \subset U$ with $x, y \in \mathfrak{l}$, i.e., there exists some k-rational map $f : \mathbb{P}^1_k \dashrightarrow U$ with $x, y \in \mathrm{im}(f) =: \mathfrak{l}$. It is clear that being rationally connected is a *birational property,* i.e., if X and Y are birationally equivalent k-varieties, then X is rationally connected if and only if Y is so. (Indeed, X and Y are birationally equivalent if and only if there exist Zariski open dense subsets $U \subset X$ and $V \subset Y$ such that U and V are k-isomorphic, etc.) In particular, being rationally connected depends on the function field $k(X)$ of X only, and not on the given X itself. It is nevertheless not known how to express rational connectedness in terms of function fields, and this goes to the core of Mori's Classification Program.

We recall that a k-variety X is called rational, if X is birationally equivalent to $\mathbb{A}^d_k$, where $d = \dim(X)$, or equivalently, $k(X) = k(t_1, \dots, t_d)$ is the rational function field in $d = \dim(X)$ variables over k. Further, a k-variety is called unirational, if X is dominated by a rational k-variety, i.e., if there exists a dominant k-rational map $\mathbb{A}^n_k \dashrightarrow X$ for some n. Concerning the relationship between the three classes of k-varieties introduced above one has the following.

Remark 7.1. In the above notations and context, the following hold:

1) The rational varieties are rationally connected, because any two closed points x, y in $\mathbb{A}^d_k$ are connected by a canonical line $\mathfrak{l}_{xy} \subset \mathbb{A}^d_k$.

2) The unirational k-varieties which are not rational are quite abundant. This goes back to the famous *Noether Problem,* which asks the following: Let $G \hookrightarrow \mathrm{GL}(V)$ be an n-dimensional linear representation of a finite group G. Is then the fixed field $k(V)^G$ a rational function field ? A more special variant of this question is: Let the permutation group S_n act on the rational function field $k(t_1, \dots, t_n)$ in the usual way, and $G \hookrightarrow S_n$ be a subgroup. Is then the fixed field $k(t_1, \dots, t_n)^G$ a rational function field in n variables?

- The answer to Noether's Problem in general is negative, thus in geometric terms one has that $X := G\backslash\mathbb{A}^n_k$ is a unirational k-variety which is not rational.

- We finally notice that it is a highly non-trivial and fundamental open problem to characterize (say, in terms of cohomological invariants) the rational k-varieties among the unirational ones.

3) Given a morphism $\varphi : Y \to X$ of k-varieties, and a line $\mathfrak{l} \subset X$, it follows that $\varphi(\mathfrak{l}) \subset Y$ is either a point or a line. In particular, the unirational varieties are rationally connected, and both the unirational as well as the rationally connected k-varieties are stable under dominant k-rational maps. One has implications:

$$\text{rational} \Rightarrow \text{unirational} \Rightarrow \text{rationally connected}$$

It is nevertheless one of the major open problems of Mori's Classification Problems to show that rationally connected varieties are actually unirational.

4) Finally, the class of uniruled k-varieties strictly contains the rationally connected varieties, because k-varieties of the form $X := \mathbb{P}^1_k \times Y$ are uniruled, but if Y is not rationally connected, e.g., if Y is a curve of positive geometric genus, then X is uniruled, but not rationally connected.

Before mentioning rationality conditions in the context of rationally connected varieties, we mention the following characterization result, see Kollár–Miaoka–Miata [38], [39]:

Fact 7.2. Let k be an algebraically closed field with $\mathrm{char}(k) = 0$, and X be a complete smooth k-variety. Then the following are equivalent:

a) X is rationally connected.

b) For each pair of closed points $x, y \in X$ there exists a line $\mathfrak{l} \subset X$ with $x, y \in \mathfrak{l}$.

c) Every finite set of closed points $S \subset X$ is contained in a line $\mathfrak{l} \subset X$.

d) There exists $f : \mathbb{P}^1_k \to X$ such that f^*T_X is ample on $\mathbb{P}^1_k$.

e) For every finite set of closed points $S \subset X(k)$ there exists $f : \mathbb{P}^1_k \to X$ such that f^*T_X is ample on $\mathbb{P}^1_k$ and $S \subset \mathrm{im}(f)$.

If $\mathrm{char}(k) > 0$, the situation is somewhat more complicated, because the conditions a)-e) given above are not equivalent. Nevertheless, it turns out that condition d) implies the rest, see Kollár [36], IV.3.9. Varieties which satisfy condition d) are called separably rationally connected. Unfortunately, this notion is *stronger than being unirational.* Using this equivalent form of the rational connectedness, Kollár [37] proves the following:

Theorem 7.3. *Let k be a large field, and X be a smooth proper k-variety such that the base change $\overline{X} := X \times_k \overline{k}$ of X to the algebraic closure $\overline{k}$ of k is a separably rationally connected $\overline{k}$-variety. Then, for every k-rational point $x \in X(k)$, there is a k-morphism $f : \mathbb{P}^1_k \to X$ such that f^*T_X is ample on $\mathbb{P}^1_k$ and $x \in f\big(\mathbb{P}^1_k(k)\big)$.*

An interesting corollary of the above theorem is the following source of examples where one can prove that being rationally connected implies unirationality, see Kollár [37]:

Theorem 7.4. *Let k be a large field with $\mathrm{char}(k) = 0$, and X be a smooth proper k-variety. Assume that there exists a dominant k-morphism $\varphi : X \to \mathbb{P}^1_k$ whose geometric generic fiber $X_{\overline{\eta}}$ satisfies one of the following:*

a) *$X_{\overline{\eta}}$ is either a degree ≥ 2 Del Pezzo surface, or a cubic hypersurface, or a complete intersection of two quadrics in $\mathbb{P}^N_k$ for some $N \geq 4$.*

b) *There is a connected linear algebraic group acting on $X_{\overline{\eta}}$ with a dense orbit.*

Then X is unirational over k if and only if $X(k)$ is non-empty.

Thanks: I would like to thank D. Harbater, M. Jarden, J. Koenigsmann, F.-V. Kuhlmann, and many others for inspiration and discussions. Last but not least, many thanks to the referee and to F.-V. Kuhlmann for the careful reading of the manuscript, as well as for comments and suggestions (which went beyond a long list of typos).

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