Pro-p hom-form of the birational anabelian conjecture over sub-p-adic fields

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Abstract. We prove a Hom-form of the pro-p birational anabelian conjecture for function fields over sub-p-adic fields. Our starting point is the corresponding Theorem of Mochizuki in the case of transcendence degree 1.

1. Introduction

Let k be a fixed sub-p-adic field, i.e., a subfield of a function field over $\mathbb{Q}_p$. Fix $\overline{k}$, an algebraic closure of k, and denote by G_k the absolute Galois group. Let $\mathcal{F}_k$ be the category of regular function fields $K|k$, and k-embeddings of function fields. Further, let $\mathcal{G}_k$ be the category of profinite groups, G, endowed with a surjective augmentation morphism, $\pi_G : G \to G_k$, such that $\ker(\pi_G)$ is a pro-p group, and outer open G_k-homomorphisms. I.e., a morphism from G to H in $\mathcal{G}_k$ is of the form $\mathrm{Inn}_{G_k}(H) \circ f$, where $f : G \to H$ is an open homomorphism such that $\pi_G = \pi_H \circ f$, and $\mathrm{Inn}_{G_k}(H)$ denotes the group of inner automorphisms of H which lie over G_k. Since G_k has trivial center, $\mathrm{Inn}_{G_k}(H)$ consists exactly of the inner automorphisms of H defined by an element of $\ker(\pi_H)$. Finally, we remark that there exists a naturally defined functor from $\mathcal{F}_k$ to $\mathcal{G}_k$: for $K|k$ from $\mathcal{F}_k$, let $\tilde{K}|K\overline{k}$ be a maximal pro-p extension. Then $\tilde{K}|K$ is Galois, and $\Pi_K := \mathrm{Gal}(\tilde{K}|K)$ endowed with the projection $\pi_K : \Pi_K \to G_k$ is an object of $\mathcal{G}_k$. Further, a morphism $\imath : K|k \hookrightarrow L|k$ in $\mathcal{F}_k$ extends uniquely to a $\overline{k}$-embedding $\bar{\imath} : K\overline{k} \hookrightarrow L\overline{k}$, and $\bar{\imath}$ has prolongations $\tilde{\imath} : \tilde{K} \hookrightarrow \tilde{L}$. Each such prolongation $\tilde{\imath} : \tilde{K} \hookrightarrow \tilde{L}$ gives rise to an open G_k-homomorphism $\Phi_{\tilde{\imath}} : \Pi_L \to \Pi_K$ defined by

$$\Phi_{\tilde{\imath}}(g) = \tilde{\imath}^{-1} \circ g \circ \tilde{\imath}, \quad g \in \Pi_L\,,$$

and any two such prolongations are conjugate by an element from $\ker(\pi_K)$. Thus, sending each $K|k$ from $\mathcal{F}_k$ to $\pi_K : \Pi_K \to G_k$ in $\mathcal{G}_k$ yields a well defined functor from $\mathcal{F}_k$ to $\mathcal{G}_k$.

The purpose of this note is to prove the following *Galois by pro-p Hom-form* of the birational anabelian conjecture:

Theorem 1 *The above functor from $\mathcal{F}_k$ to $\mathcal{G}_k$ is fully faithful, i.e., for regular function fields $K|k$ and $L|k$, there is a canonical bijection*

$$\mathrm{Hom}_k(K, L) \to \mathrm{Hom}_{\mathcal{G}_k}(\Pi_L, \Pi_K).$$

Equivalently, for fixed field extensions $\tilde{K}|K$ and $\tilde{L}|L$ as above, the map

$$\tilde{\imath} \mapsto \Phi_{\tilde{\imath}} \quad \textit{with} \;\; \Phi_{\tilde{\imath}}(g) = \tilde{\imath}^{-1} \circ g \circ \tilde{\imath} \;\; \textit{for} \;\; g \in \Pi_L \,,$$

is a bijection from the set of $\overline{k}$-embeddings $\tilde{\imath} : \tilde{K} \hookrightarrow \tilde{L}$ onto the set of all the open G_k-morphisms $\Pi_L \to \Pi_K$.

Before embarking on the proof, the following comments are in order: first, if $\mathrm{td}(K|k) = 1$, then Theorem 1 is a special case of Theorem 16.5 in the fundamental paper by Mochizuki [2]. Second, the above Theorem 1 implies the corresponding *full profinite version,* in which Π_K is replaced by the full absolute Galois group G_K of K. But naturally, the above Theorem 1 does not follow from the corresponding full profinite version. Finally, the full profinite version of Theorem 1 above was proved by Mochizuki in loc. cit., where it appears as Corollary 17.1. There he uses an inductive procedure on $\mathrm{td}(K|k)$, which is ill-suited to the pro-p situation, and hence he obtains only a profinite result. In our proof of Theorem 1 we use Mochizuki's pro-p result for the transcendence degree one case, but instead of proceeding inductively on the transcendence degree, we will make use of the second author's ideas as described in [3].

2. Proof of Theorem 1

The proof of Theorem 1 will have two parts:

i) Given an open G_k-homomorphism $\Phi : \Pi_L \to \Pi_K$, there exists a $\overline{k}$-embedding $\tilde{\imath}_\Phi : \tilde{K} \hookrightarrow \tilde{L}$ which defines Φ as indicated above, i.e., such that $\Phi = \Phi_{\tilde{\imath}_\Phi}$.
ii) The map $\tilde{\imath} \mapsto \Phi_{\tilde{\imath}}$ is injective.

We begin by recalling some basic facts from Kummer Theory. First, for every $K|k$ from $\mathcal{F}_k$, let $\mathbb{T}_{p,K} = \varprojlim_e \mu_{p^e,K}$ be the p-adic Tate module of $\mathbb{G}_{m,K}$. Then via the canonical projection $\pi_K : \Pi_K \to G_k$, we can/will identify $\mathbb{T}_{p,K}$ with $\mathbb{T}_{p,k}$, and denote it simply by $\mathbb{T}_p$. Second, let $\widehat{K}$ denote the p-adic completion of $K^\times$, and $\jmath_K : K^\times \to \widehat{K}$

the completion morphism. We note that $\ker(\jmath_K)$ is the maximal p^∞-divisible subgroup of $K^\times$, hence by the structure of $K|k$ it follows that $\ker(\jmath_K) = \mu'$ is the group of roots of unity in k of order prime to p. Further, $K^\times/k^\times$ is a free abelian group, hence it injects into its p-adic completion $(K^\times/k^\times)\widehat{\ }$ which itself equals $\widehat{K}/\widehat{k}$. Next, recall that Kummer Theory yields a canonical ismorphism of p-adically complete groups $\delta_K : \widehat{K} \to \mathrm{H}^1(\Pi_K, \mathbb{T}_p)$. Therefore we will make the identification $\widehat{K} = \mathrm{H}^1(\Pi_K, \mathbb{T}_p)$, if this does not lead to confusion. By the functoriality of Kummer Theory, the surjective projection $\pi_K : \Pi_K \to G_k$ gives rise to an embedding $\mathrm{H}^1(\pi_K) : \widehat{k} \hookrightarrow \widehat{K}$ which is nothing but the p-adic completion of the structural morphism $k \hookrightarrow K$. Furthermore, if $K|k$ and $L|k$ are objects from $\mathcal{F}_k$, and $\Phi : \Pi_L \to \Pi_K$ is an open G_k-morphism, then by functoriality we get an embedding of p-adically complete groups

$$\mathrm{H}^1(\Phi) : \widehat{K} = \mathrm{H}^1(\Pi_K, \mathbb{T}_p) \to \mathrm{H}^1(\Pi_L, \mathbb{T}_p) = \widehat{L}$$

which identifies $\widehat{k} \subset \widehat{K}$ with $\widehat{k} \subset \widehat{L}$. Finally note that if $\Phi = \Phi_{\tilde{\imath}}$ is defined by a morphism $\imath : K|k \hookrightarrow L|k$ from $\mathcal{F}_k$, then $\mathrm{H}^1(\Phi_{\tilde{\imath}}) = \widehat{\imath}$ is nothing but the p-adic completion of the k-embedding $\imath : K|k \hookrightarrow L|k$, and therefore one has:

$$(\dagger) \qquad \mathrm{H}^1(\Phi_{\tilde{\imath}}) \circ \jmath_K(x) = \jmath_L \circ \imath(x), \quad x \in K^\times.$$

Proof of i):

Claim 1. $\mathrm{H}^1(\Phi) \circ \jmath_K(K^\times) \subseteq \jmath_L(L^\times)$.

Proof of Claim 1: Consider $t \in K^\times$ arbitrary. First, if $t \in k^\times$, then $\mathrm{H}^1(\Phi)$ identifies $\jmath_K(t)$ with $\jmath_L(t)$ by the discussion above. Second, let $t \in K^\times \backslash k^\times$. Then the inclusion $k(t) \subseteq K$ is a morphism in $\mathcal{F}_k$, hence gives rise canonically to an open G_k-morphism $\Phi_t : \Pi_K \to \Pi_{k(t)}$. But then $\Phi_t \circ \Phi : \Pi_L \to \Pi_{k(t)}$ is an open G_k-homomorphism too. Since $\mathrm{td}\big(k(t)|k\big) = 1$, it follows by Theorem 16.5 from [2] that $\Phi_t \circ \Phi$ is defined by a k-embedding $\imath_t : k(t) \to L$, i.e., of the form $\Phi_t \circ \Phi = \Phi_{\tilde{\imath}_t}$. Hence by the assertion $(\dagger)$ above, $\mathrm{H}^1(\Phi_t \circ \Phi) : \widehat{k(t)} \to \widehat{L}$ is exactly the p-adic completion of $\imath_t : k(t) \to L$, and we get:

$$\mathrm{H}^1(\Phi_t \circ \Phi) \circ \jmath_{k(t)}(x) = \jmath_L \circ \imath_t(x) \in \jmath_L(L^\times), \quad x \in k(t)^\times.$$

By functoriality we have $\mathrm{H}^1(\Phi_t \circ \Phi) = \mathrm{H}^1(\Phi) \circ \mathrm{H}^1(\Phi_t)$, and $\mathrm{H}^1(\Phi_t)$ is the p-adic completion of the inclusion $k(t) \subseteq K$, hence $\mathrm{H}^1(\Phi_t) \circ \jmath_{k(t)}(x) = \jmath_K(x)$ for $x \in k(t)^\times$. Combining these equalities, we finally get

$$(\dagger') \qquad \mathrm{H}^1(\Phi) \circ \jmath_K(x) = \jmath_L \circ \imath_t(x) \in \jmath_L(L^\times), \quad x \in k(t)^\times.$$

Since t was arbitrary, this concludes the proof of Claim 1. □

Next let us identify $K^\times/\mu'$ and $L^\times/\mu'$ with their images in $\widehat{K}$, respective $\widehat{L}$, via the p-adic completion homomorphisms $\jmath_K$, respectively $\jmath_L$. Then by Claim 1 above, $\mathrm{H}^1(\Phi)$ maps $K^\times/\mu'$ into $L^\times/\mu'$, and identifies $k^\times/\mu' \subset K^\times/\mu'$ with $k^\times/\mu' \subset L^\times/\mu'$. Further, $\mathrm{H}^1(\Phi)$ is nothing but the p-adic completion of its restriction to $K^\times/\mu'$. Modding out by $k^\times/\mu'$ thus yields an embedding of free abelian groups $\alpha : K^\times/k^\times \hookrightarrow L^\times/k^\times$ canonically defined by $\mathrm{H}^1(\Phi)$. Thinking now of $(K,+)$ and $(L,+)$ as infinite dimensional k-vector spaces, $\mathcal{P}(K) := K^\times/k^\times$ and $\mathcal{P}(L) := L^\times/k^\times$ are their projectivizations. Then $\alpha : \mathcal{P}(K) \hookrightarrow \mathcal{P}(L)$ is an inclusion which respects the multiplicative structures.

Claim 2. The map $\alpha : \mathcal{P}(K) \hookrightarrow \mathcal{P}(L)$ preserves collineations.

Proof of Claim 2: A line in $\mathcal{P}(K)$ is the image of a two-dimensional k-subspace of K, say $\mathfrak{l}_{t_1t_2} := kt_1 + kt_2$, where $t_1, t_2 \in K$ are k-linearly independent. Note that $\mathfrak{l}_{t_1t_2} = t_1 \cdot \mathfrak{l}_t$, where $t = t_2/t_1$ and $\mathfrak{l}_t := k + kt$. Since multiplication by $t_1/k^\times$ is a line-preserving automorphism of $K^\times/k^\times = \mathcal{P}(K)$, α is multiplicative, and multiplication by $\alpha(t_1/k^\times)$ is a line-preserving automorphism of $\mathcal{P}(L)$, it suffices to show that α maps the lines $\mathfrak{l}_t$, $t \in K\backslash k$, to lines in $\mathcal{P}(L)$. In order to do this, remark that by relation $(\dagger')$ above we have:

$$\mathrm{H}^1(\Phi) \circ \jmath_K(kt+k)^\times = \jmath_L \circ \imath_t(kt+k)^\times = \jmath_L\big(k\imath_t(t)+k\big)^\times.$$

Thus $\mathfrak{l}_t \subset \mathcal{P}(K)$ is mapped bijectively onto $\mathfrak{l}_{\imath_t(t)} \subset \mathcal{P}(L)$. $\square$

Let $L_0^\times \subseteq L^\times$ denote the preimage of $\alpha\big(\mathcal{P}(K)\big)$ in L, and let us set $L_0 := L_0^\times \cup \{0_L\} \subseteq L$. Since α is collineation preserving, it follows that L_0 is a field containing k, and $\alpha : \mathcal{P}(K) \to \mathcal{P}(L_0)$ is a collineation preserving bijection. By the Fundamental Theorem of projective geometry, see Artin [1], we conclude that α is induced by an isomorphism of k-vector spaces $\alpha_K : (K,+) \to (L_0,+)$, which is unique up to k-semilinear homotheties. As in [3], it follows that setting $\imath_K := \alpha(1_K)^{-1} \cdot \alpha_K$, the resulting map $\imath_K : K \to L$ is actually a k-isomorphism of fields, whose projectivization equals α. In particular, the p-adic completion of $\imath_K$ equals $\mathrm{H}^1(\Phi)$, and $\imath_K$ is the unique embedding of fields $K \hookrightarrow L$ with this property.

Now let $K'|K$ be a finite Galois extension contained in $\tilde{K}$, and $k' := K' \cap \overline{k}$. Then $\Pi_{K'}$ is an open normal subgroup of Π_K, and $\Phi^{-1}(\Pi_{K'}) = \Pi_{L'}$ for some finite Galois extension $L'|L$ contained in $\tilde{L}$. Since Φ is a G_k-homomorphism, it follows that $L' \cap \overline{k} = k'$, and $K'|k'$ and $L'|k'$ are regular function fields over k' which is a sub-p-adic field. The restriction Φ' of Φ to $\Pi_{L'}$ yields an open $G_{k'}$-homomorphism $\Phi' : \Pi_{L'} \to \Pi_{K'}$. *Mutatis mutandis,* we obtain from Φ' a k'-embedding $\imath_{K'} : K' \hookrightarrow L'$ such that its p-adic completion is $\mathrm{H}^1(\Phi')$. The compatibility relation $\mathrm{res}^{\Pi_{L'}}_{\Pi_L} \circ \mathrm{H}^1(\Phi') = \mathrm{H}^1(\Phi) \circ \mathrm{res}^{\Pi_{K'}}_{\Pi_K}$ translates into the fact that $\imath_K$ is the restriction of $\imath_{K'}$ to K. Note that extensions of the form $K'|K$ above exhaust $\tilde{K}|K$, so taking limits we obtain a $\overline{k}$-embedding $\tilde{\imath}_\Phi : \tilde{K} \hookrightarrow \tilde{L}$ such that $(\tilde{\imath}_\Phi)|_{K'} = \imath_{K'}$ for all $K'|K$.

Claim 3. $\Phi_{\tilde{\imath}_\Phi} = \Phi$.

Proof of Claim 3: Indeed, for $K'|K$ and the corresponding $L'|L$ as above, Φ yields a surjection of Galois groups

$$\Psi' : \mathrm{Gal}(L'|L) \to \mathrm{Gal}(K'|K)\,.$$

Note that $\mathrm{H}^1(\Pi_{K'}, \mathbb{T}_p)$ is a $\mathrm{Gal}(K'|K)$ module, and correspondingly for $L'|L$. Moreover, for all $\sigma \in \mathrm{Gal}(L'|L)$ we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}^1(\Pi_{K'}, \mathbb{T}_p) & \xrightarrow{\mathrm{H}^1(\Phi')} & \mathrm{H}^1(\Pi_{L'}, \mathbb{T}_p) \\
{\scriptstyle \Psi'(\sigma)}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\
\mathrm{H}^1(\Pi_{K'}, \mathbb{T}_p) & \xrightarrow{\mathrm{H}^1(\Phi')} & \mathrm{H}^1(\Pi_{L'}, \mathbb{T}_p).
\end{array}$$

which via the Kummer Theory isomorphisms translates into:

$$\imath_{K'} \circ \Psi'(\sigma) = \sigma \circ \imath_{K'}, \quad \sigma \in \mathrm{Gal}(K'|K).$$

Hence taking limits over all the $K'|K$, we get as required:

$$\tilde{\imath}_\Phi \circ \Phi(g) = g \circ \tilde{\imath}_\Phi, \quad g \in \Pi_L\,. \ \square$$

Proof of ii): Suppose we are given a $\overline{k}$-embedding $\tilde{\imath} : \tilde{K} \hookrightarrow \tilde{L}$, and let $\Phi_{\tilde{\imath}} : \Pi_L \to \Pi_K$ be the corresponding open G_k-homomorphism. For every finite Galois sub-extension $L'|L$ of $\tilde{L}|L$ and the corresponding $K' := \tilde{\imath}^{-1}(L')$, the restriction of $\Phi_{\tilde{\imath}}$ to $\Pi_{L'}$ defines an open $G_{k'}$-homomorphism $\Phi_{\tilde{\imath},L'} : \Pi_{L'} \to \Pi_{K'}$, where $k' := \overline{k} \cap L'$. By the discussions above, the Kummer morphism $\mathrm{H}^1(\Phi_{\tilde{\imath},L'})$ is the p-adic completion of $\imath_{K'} := \tilde{\imath}|_{K'}$, and in particular, $\mathrm{H}^1(\Phi_{\tilde{\imath},L'})$ dermines $\imath_{K'}$ uniquely. By taking limits, it follows that $\tilde{\imath}$ is uniquely determined by the family of Kummer morphisms $\mathrm{H}^1(\Phi_{\tilde{\imath},L'})$ with $L'|L$ as above. Assertion ii) is thus proven. $\square$

This concludes the proof of Theorem 1.

References

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