# Chapter 10
# $\mathbb{Z}/\ell$ abelian-by-central Galois theory of prime divisors

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**Abstract** In this manuscript I show how to recover some of the inertia structure of (quasi) divisors of a function field $K|k$ over an algebraically closed base field $k$ from its maximal mod $\ell$ *abelian-by-central* Galois theory of $K$, provided $\mathrm{td}(K|k) > 1$. This is a first technical step in trying to extend Bogomolov's birational anabelian program beyond the full pro-$\ell$ situation, which corresponds to the limit case mod $\ell^\infty$.

## 10.1 Introduction

At the beginning of the 1990's, Bogomolov [Bo91] initiated a program whose final aim is to recover function fields $K|k$ over algebraically closed base fields $k$ from their pro-$\ell$ abelian-by-central Galois theory. That program goes beyond Grothendieck's birational anabelian program as initiated in [Gro83], [Gro84], because $k$ being algebraically closed, there is no arithmetical Galois action in the game. In a few words, the precise context for Bogomolov's birational anabelian program is as follows:

- Let $\ell$ be a fixed rational prime number.
- Consider function fields $K|k$ with $k$ algebraically closed of characteristic $\neq \ell$.
- Let $K' \hookrightarrow K''$ be maximal pro-$\ell$ abelian, respectively *abelian-by-central,* extensions of $K$.
- Let $pr : \Pi^{\mathrm{c}}_K \to \Pi_K$ be the corresponding projection of Galois groups.

Notice that $pr : \Pi^{\mathrm{c}}_K \to \Pi_K$ can be recovered group theoretically from $\Pi^{\mathrm{c}}_K$, as its kernel is exactly the topological closure of the commutator subgroup of $\Pi^{\mathrm{c}}_K$. Actually, if $G^{(1)} = G_K$, and for $i \geq 1$ we let

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$$G^{(i+1)} := [G^{(i)}, G^{(1)}](G^{(i)})^{\ell^\infty}$$

be the closed subgroup of $G^{(i)}$ generated by all the commutators $[x,y]$ with $x \in G^{(i)}$, $y \in G^{(1)}$ and the $\ell^\infty$-powers of all the $z \in G^{(i)}$, then the $G^{(i)}$, $i \geq 1$, are the descending central $\ell^\infty$ terms of the absolute Galois group $G_K$, and

$$\Pi_K^{\mathrm{c}} = G^{(1)}/G^{(3)} \to \Pi_K = G^{(1)}/G^{(2)}.$$

Further, denoting by $G^{(\infty)}$ the intersection of all the $G^{(i)}$, it follows that $G_K(\ell) := G_K/G^{(\infty)}$ is the maximal pro-$\ell$ quotient of $G_K$, see e.g. [NSW08] page 220. The Program initiated by Bogomolov[2] mentioned above has as ultimate goal to recover function fields $K|k$ as above from $\mathrm{Gal}(K''|K)$ in a functorial way. If completed, this Program would go far beyond Grothendieck's birational anabelian geometry, see [Pop98] for a historical note on birational anabelian geometry, and [PopXX] Introduction, for a historical note on Bogomolov's Program and an outline of a strategy to tackle this Program; and for an early history beginning even before [Gro83], [Gro84], see [NSW08] Chapter XII, the original sources [Ne69], [Uch79], as well as Szamuely's Séminaire Bourbaki talk [Sz04]. To conclude, I would like to mention that in contrast to Grothedieck's birational anabelian program — which is completed to a large extent — Bogomolov's birational anabelian program is completed only in the case the base field $k$ is an *algebraic closure of a finite field*, see Bogomolov–Tschinkel [BT08] for the case $\mathrm{td}(K|k) = 2$ and Pop [PopXX] in general.

The results of the present manuscript represent a first step and hints at the possibility that a mod $\ell$ *abelian-by-central* form of birational anabelian geometry might hold, which would then go beyond Bogomolov's birational anabelian program in many ways.

In order to put the results of this paper in the right perspective, let me mention that the present paper shares similarities with [Pop06] and [Pop10], where similar results were obtained, but working with the full pro-$\ell$ Galois group $G_K(\ell)$, respectively the maximal pro-$\ell$ abelian-by-central Galois group $\Pi_K^{\mathrm{c}}$. Whereas a key technical tool used in [Pop10] is the theory of $\mathbb{Z}_\ell$ "commuting liftable pairs" as developed in Bogomolov–Tschinkel [BT02] — which was used as a "black box" — we use here the theory of $\mathbb{Z}/\ell$ commuting liftable pairs. The fact that such a mod $\ell$ variant might hold was already suggested by previous results from Mahé–Mináč–Smith [MMS04] in the case $\ell = 2$. The details for the theory of $\mathbb{Z}/\ell$ commuting liftable pairs can be found in the manuscript Topaz [ToXX].

Before going into the details of the manuscript, let me introduce notations which will used throughout the manuscript and mention briefly facts introduced later on.

- Let $\ell$ be a prime number.
- Consider function fields $K|k$ with $k$ algebraically closed of characteristic $\neq \ell$.
- Let $K' \hookrightarrow K''$ be a maximal $\mathbb{Z}/\ell$ abelian extension, respectively a maximal $\mathbb{Z}/\ell$ abelian-by-central extension of $K$.
- Let $pr : \overline{\Pi}_K^{\mathrm{c}} \to \overline{\Pi}_K$ be the corresponding quotient map of Galois groups.

[2] Recall that Bogomolov denotes $\mathrm{Gal}(K''|K)$ by $\mathrm{PGal}_K^{\mathrm{c}}$.

Notice that $pr : \overline{\Pi}^{c}_{K} \to \overline{\Pi}_{K}$ can be recovered group theoretically from $\overline{\Pi}^{c}_{K}$, as its kernel is the (topological closure of the) commutator group of $\overline{\Pi}^{c}_{K}$.

Let $v$ be a valuation of $K$, and $v'$ some prolongation of $v$ to $K'$. Let $T_{v'} \subseteq Z_{v'}$ be the inertia group, respectively decomposition group, of $v'$ in $\overline{\Pi}_{K}$. By Hilbert decomposition theory for valuations, the groups $T_{v'} \subseteq Z_{v'}$ of the several prolongations $v'$ of $v$ to $K'$ are conjugated. Thus, since $\overline{\Pi}_{K}$ is abelian, the groups $T_{v'} \subseteq Z_{v'}$ depend on $v$ only, and not on its prolongations $v'$ to $K'$. We will denote these groups by $T_v \subseteq Z_v$, and call them the inertia group, respectively decomposition group, at $v$. We also notice that the residue field $K'v'$ of $v'$ is actually a maximal $\mathbb{Z}/\ell$ abelian extension $K'v' = (Kv)'$ of the residue field $Kv$ of $v$, see Fact 2 (3) below.

We next recall that for a $k$-valuation of $K$, i.e., a valuation of $K$ whose valuation ring $\mathcal{O}_v$ contains $k$, and thus $k$ canonically embeds in the residue field $Kv := \mathcal{O}_v/\mathbf{m}_v$, the following conditions are equivalent:

(i) The valuation ring $\mathcal{O}_v$ equals the local ring $\mathcal{O}_{X,x_v}$ of the generic point $x_v$ of some Weil prime divisor of some normal model $X \to \mathrm{Spec}(k)$ of $K|k$.
(ii) The transcendence degrees satisfy $\mathrm{td}(Kv\,|\,k) = \mathrm{td}(K|k) - 1$.

A **prime divisor** of $K|k$ is any $k$-valuation $v$ of $K$ which satisfies the above equivalent conditions. In particular, if $v$ is a prime divisors of $K|k$, then $vK \cong \mathbb{Z}$ and $Kv|k$ is a function field satisfying $\mathrm{td}(Kv\,|\,k) = \mathrm{td}(K|k) - 1$. By Hilbert decomposition theory for valuations, see e.g. [Bou64] Ch. 6, it follows that the following hold:

$$T_v \cong \mathbb{Z}/\ell \quad \text{and} \quad Z_v \cong T_v \times \mathrm{Gal}(K'v'|Kv) \cong \mathbb{Z}/\ell \times \mathrm{Gal}(K'v'|Kv).$$

For a prime divisor $v$, we will call $Z_v$ endowed with $T_v$ a **divisorial subgroup** of $\overline{\Pi}_{K}$ or of the function field $K|k$.

As a first step in recovering $K|k$ from its mod $\ell$ abelian-by-central Galois theory, one would like to recover the divisorial subgroups of $\overline{\Pi}_{K}$ from $\overline{\Pi}^{c}_{K}$. This is indeed possible if $k$ is the algebraic closure of a finite field, see below. Unfortunately, there are serious difficulties when one tries to do the same in the case $k$ is not an algebraic closure of a finite field, as the non-trivial valuations of $k$ interfere. Therefore one is led to considering the following generalization of prime divisors, see e.g. [Pop06] Appendix: a valuation $v$ of $K$ is called **quasi divisorial**, or a **quasi prime divisor** of $K$, if the valuation ring $\mathcal{O}_v$ of $v$ is maximal among the valuation rings of valuations of $K$ satisfying:

(i) The relative value group $vK/vk$ is isomorphic to $\mathbb{Z}$ as abstract groups.
(ii) The residue extension $Kv\,|\,kv$ is a function field with $\mathrm{td}(Kv\,|\,kv) = \mathrm{td}(K|k) - 1$.

Notice that a quasi prime divisor $v$ of $K$ is a prime divisor if and only if $v$ is trivial on $k$. In particular, in the case where $k$ is an *algebraic closure of a finite field,* the quasi prime divisors and the prime divisors of $K|k$ coincide (as all valuations of $K$ are trivial on $k$).

For a Galois extension $\tilde{K}|K$ and its Galois group $\mathrm{Gal}(\tilde{K}|K)$, we will say that a subgroup $Z$ of $\mathrm{Gal}(\tilde{K}|K)$ endowed with a subgroup $T$ of $Z$ is a **quasi-divisorial subgroup** of $\mathrm{Gal}(\tilde{K}|K)$ — or of $K|k$ in case $\mathrm{Gal}(\tilde{K}|K)$ is obvious from the context —

if $T \subseteq Z$ are the inertia group, respectively the decomposition group, above some quasi-divisor $v$ of $K|k$.

It was the main result in Pop [Pop10] to show that the quasi-divisorial subgroups of the Galois group $\Pi_K$ can be recovered by a group theoretical recipe from the canonical projection $\Pi_K^{\rm c} \to \Pi_K$, which itself can be recovered from $\Pi_K^{\rm c}$. Paralleling that result, the main results of this paper can be summarized as follows:

**Theorem 1.** *Let $K|k$ be a function field over the algebraically closed field $k$ of characteristic $\mathrm{char}(k) \neq \ell$, and let $\overline{\Pi}_K^{\rm c} \to \overline{\Pi}_K$ be the canonical projection. For subgroups $T$, $Z$, $\Delta$ of $\overline{\Pi}_K$, let $T''$, $Z''$, $\Delta''$ denote their preimages in $\overline{\Pi}_K^{\rm c}$. Then one has:*

*(1) The transcendence degree $d = \mathrm{td}(K|k)$ is the maximal integer $d$ such that there exist closed subgroups $\Delta \cong (\mathbb{Z}/\ell)^d$ of $\overline{\Pi}_K$ with $\Delta''$ abelian.*
*(2) Suppose that $d := \mathrm{td}(K|k) > 1$. Let $T \subset Z$ be closed subgroups of $\overline{\Pi}_K$. Then $Z$ endowed with $T$ is a quasi divisorial subgroup of $\overline{\Pi}_K$ if and only $Z$ and $T$ are maximal in the set of closed subgroups of $\overline{\Pi}_K$ which satisfy:*

   *(i) $Z$ contains a closed subgroup $\Delta \cong (\mathbb{Z}/\ell)^d$ such that $\Delta''$ is abelian.*
   *(ii) $T \cong \mathbb{Z}/\ell$, and $T''$ is the center of $Z''$.*

Actually the above Theorem is a special case of the more general assertions Proposition 15, and Theorems 17 and 19, which deal with generalized [almost] (quasi) prime $r$-divisors. The above Theorem corresponds to the case $r = 1$.

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## 10.2 Basic facts from valuation theory

### *10.2.1 Hilbert decomposition in abelian extensions*

Let $K$ be a field of characteristic $\neq \ell$ containing all $\ell$-th roots of unity $\mu_\ell$. In particular, we can fix a (non-canonically) isomorphism $\imath_K : \mu_\ell \to \mathbb{Z}/\ell$ as $G_K$ modules. Kummer theory provides a canonical non-degenerate pairing

$$K^\times/\ell \times \overline{\Pi}_K \to \mu_\ell\,,$$

that by Pontrjagin duality gives rise to canonical isomorphisms $\overline{\Pi}_K = \mathrm{Hom}(K^\times, \mu_\ell)$ and $K^\times/\ell = \mathrm{Hom}_{\rm cont}(G_K, \mu_\ell)$. Thus we finally get isomorphisms:

$$\overline{\Pi}_K = \mathrm{Hom}(K^\times, \mu_\ell) \xrightarrow{\imath_K} \mathrm{Hom}(K^\times, \mathbb{Z}/\ell).$$

In the above context, let $v$ be a valuation of $K$, and $v'$ some prolongation of $v$ to $K'$. Suppose that the residual characteristic $p = \mathrm{char}(Kv) \neq \ell$. We denote by $T_v \subseteq Z_v$ the inertia, respectively decomposition, groups of $v'|v$ in $\overline{\Pi}_K$. We notice that the ramification group $V_v$ of $v'|v$ is trivial, as $p = \mathrm{char}(Kv) \neq \ell$ does not divide the order of $\overline{\Pi}_K$, thus that of $Z_v$. Furthermore, $T_v \subseteq Z_v$ depends only on $v$, as $K'|K$ is abelian. Finally, we denote by $K^T$ and $K^Z$ the corresponding fixed fields in $K'$.

The following are well known facts, and we reproduce them here for the convenience of the readers.

**Lemma 2.** *In the above context one has the following:*

*(1) Let $U_v^1 = 1 + \mathbf{m}_v \leq K^\times$ be the group of principal $v$-units in $K$. Then $K^Z|K$ is the abelian extension of $K$ obtained by adjoining the $\ell$ roots of all the elements $x \in U_v^1$, i.e., $K^Z = K[\sqrt[\ell]{U_v^1}\,]$. In particular, $K^Z = K'$ if and only if $K^\times/\ell = U_v^1/\ell$.*

*(2) Let $U_v$ be the group of $v$-units in $K$. Then $K^T|K$ is the abelian extension of $K$ obtained by adjoining the $\ell^{\mathrm{th}}$ roots of all the elements $x \in U_v$, i.e., $K^T = K[\sqrt[\ell]{U_v}\,]$. In particular:*

*(a) $K^T = K'$ if and only if $K^\times/\ell = U_v/\ell$ if and only if $K^T v' = K'v'$.*
*(b) $K^Z = K^T$ if and only if $U_v/\ell = U_v^1/\ell$ if and only if $K^Z v' = K'v'$.*

*(3) We have $K'v' = (Kv)'$, and setting $\delta_v = \dim(vK/\ell)$, the following holds:*

*(a) $T_v = \mathrm{Hom}(vK, \mathbb{Z}/\ell) \cong (\mathbb{Z}/\ell)^{\delta_v}$ non-canonically.*
*(b) $G_v := Z_v/T_v = \mathrm{Gal}\big((Kv)'|Kv\big)$.*
*(c) There are isomorphisms (the latter non-canonical) of $\ell$-torsion groups:*

$$Z_v \cong T_v \times G_v \cong (\mathbb{Z}/\ell)^{\delta_v} \times \mathrm{Gal}\big((Kv)'|Kv\big)\,.$$

*Proof.* (1) Let $K^h$ be some Henselization of $K$ containing $K^Z$. Then by general decomposition theory, $K^Z = K^h \cap K'$. We first prove that $\sqrt[\ell]{U_v^1} \subseteq K^Z$. Equivalently, if $a = 1 + x \in U_v^1$ is some principal $v$-unit, we have to show that $\sqrt[\ell]{a} \in K^Z$. Recall that by hypothesis we have: $p = \mathrm{char}(Kv) \neq \ell$, and that $K$ contains the $\ell$ roots of unity. Since $a \equiv 1 \pmod{\mathbf{m}_v}$, it follows that $X^\ell - a \equiv X^\ell - 1 \pmod{\mathbf{m}_v}$, hence $X^\ell - a$ has $\ell$ distinct roots $(\mathrm{mod}\,\mathbf{m}_v)$. By Hensel's Lemma, $X^\ell - a$ has $\ell$ distinct roots in $K^h$, i.e., $\sqrt[\ell]{a}$ is contained in $K^h$. Since $\sqrt[\ell]{a}$ is contained in $K'$ too, we finally deduce that $\sqrt[\ell]{a}$ is contained in $K^Z = K^h \cap K'$.

For the converse, consider some $a \in K^\times$ such that $\sqrt[\ell]{a} \in K^Z$. We show that $\sqrt[\ell]{a}$ is contained in $K[\sqrt[\ell]{U_v^1}\,]$. Indeed, since $K$ and $K^Z$ have equal value groups, and $\sqrt[\ell]{a} \in K^Z$, it follows that there is an element $b \in K$ such that $v\sqrt[\ell]{a} = vb$, hence $va = \ell \cdot vb$. We set $c := a/b^\ell \in U_v$ and find $\sqrt[\ell]{c} = \sqrt[\ell]{a}/b \in K^Z$. Since $K$ and $K^Z$ have equal residue fields, it follows that there is $d \in U_v$ such that $\sqrt[\ell]{c} \equiv d \bmod \mathbf{m}_v$, hence $c = d^\ell \cdot a_1$ with $a_1 \in U_v^1$. Thus finally $\sqrt[\ell]{a} = bd\sqrt[\ell]{a_1} \in K[\sqrt[\ell]{U_v^1}]$, as claimed.

The equivalence is just the translation via Kummer theory of the fact that we have equalities $K' = K[\sqrt[\ell]{K^\times}\,]$ and $K[\sqrt[\ell]{U_v^1}\,] = K^Z$.

The proof of (2) is similar, and therefore we will omit the details. And finally, (3) is just a translation in Galois terms of the assertions (1) and (2). ◻

Recall that for given valuation $v$, one can recover $\mathcal{O}_v$ from $\mathbf{m}_v$, respectively $U_v^1$, respectively $U_v$. Indeed, if $U_v$ is given, then

$$U_v^1 = \{\, x \in U_v \mid x \notin U_v - 1 \,\},$$

by which we deduce $\mathbf{m}_v = U_v^1 - 1$, and finally recover $\mathcal{O}_v$ through its complement

$$K \backslash \mathcal{O}_v = \{\, x \in K^\times \mid x^{-1} \in \mathbf{m}_v \,\}.$$

Finally, recall that for given valuations $v, w$ of $K$, with valuation rings $\mathcal{O}_v$, respectively $\mathcal{O}_w$, we say that $w \le v$, or that $w$ is a coarsening of $v$, if $\mathcal{O}_v \subseteq \mathcal{O}_w$. From the discussion above we deduce that for given valuations $v$, $w$, of $K$, the following assertions are equivalent:

(i) $w$ is a coarsening of $v$.
(ii) $\mathbf{m}_w \subseteq \mathbf{m}_v$.
(iii) $U_w^1 \subseteq U_v^1$.
(iv) $U_v \subseteq U_w$.

These facts have the following Galois theoretic translation:

**Fact 3** *In the above context and notations, the following hold:*

*(1)* $Z_v \subseteq Z_w$ *if and only if* $K_w^Z \subseteq K_v^Z$ *if and only if* $U_w^1/\ell \subseteq U_v^1/\ell$.
*(2)* $T_w \subseteq T_v$ *if and only if* $K_v^T \subseteq K_w^T$ *if and only if* $U_v/\ell \subseteq U_w/\ell$.
*(3) In particular*[3], *if* $w \le v$, *then* $T_w \subseteq T_v \subseteq Z_v \subseteq Z_w$.

### *10.2.2 The $\mathbb{Z}/\ell$ abelian form of two results of F. K. Schmidt*

In this subsection we give the abelian pro-$\ell$ form of two results of F. K. Schmidt and generalizations of these like the ones in Pop [Pop94] The local theory. See also Endler–Engler [EE77].

Let $v$ be a fixed valuation of $K$, and $v'|v$ a fixed prolongation of $v$ to $K'$. Let further $\Lambda|K$ be a fixed sub-extension of $K'|K$ containing $K^Z$. Let $\mathcal{V}'_{\Lambda,v'}$ be the set of all coarsenings $w'$ of $v'$ such that $\Lambda w' = (Kw)'$, where $w$ is the restriction of $w'$ to $K$, and let $\mathcal{V}_{\Lambda,v}$ be the restriction of $\mathcal{V}'_{\Lambda,v'}$ to $K$. We set $\mathcal{V}^0_{\Lambda,v} = \mathcal{V}_{\Lambda,v} \cup \{v\}$.

**Lemma 4.** *(1) The set* $\mathcal{V}_{\Lambda,v}$ *depends on* $v$ *and* $\Lambda$ *only, and not on the specific prolongation* $v'$ *of* $v$. *In fact,* $\mathcal{V}_{\Lambda,v}$ *consists of all the coarsenings* $w$ *of* $v$ *such that* $\Lambda w' = (Kw)'$ *for some prolongation* $w'$ *of* $w$ *to* $K'$ *(and equivalently, for every prolongation* $w'$ *of* $w$ *to* $K'$*).*

*(2) More precisely, we have* $w \in \mathcal{V}_{\Lambda,v} \iff K_w^T \subseteq \Lambda \iff \mathrm{Gal}(K'|\Lambda) \subseteq T_w$, *and in particular,* $v \in \mathcal{V}_{\Lambda,v} \iff K_v^T \subseteq \Lambda \iff \mathrm{Gal}(K'|\Lambda) \subseteq T_v$.

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[3] This is actually true for all Galois extensions $\tilde{K}|K$, and not just for $K'|K$. But then one has to start with valuations $\tilde{v}$ and coarsenings $\tilde{w}$ of those on $\tilde{K}$, etc.

*Proof.* (1) If $\tilde{v}$ is another prolongation of $v$ to $K'$, then there exists some $\sigma \in \overline{\Pi}_K$ such that $\tilde{v} = v' \circ \sigma^{-1} := \sigma(v')$, and so $\sigma$ defines a bijection $\mathcal{V}_{v',\Lambda} \to \mathcal{V}_{\tilde{v},\sigma(\Lambda)}$ by $w \mapsto w \circ \sigma^{-1}$. Note that since $\Lambda|K$ is abelian, thus in particular Galois, one has $\Lambda = \sigma(\Lambda)$. Thus for $w' \in \mathcal{V}_{\Lambda,v'}$, and $\tilde{w} = \sigma(w') := w' \circ \sigma$ one has: $\sigma$ gives rise to an $Kw$-isomorphism of the residue fields $(Kw)' = \Lambda w' \to \sigma(\Lambda)\sigma(w') = \Lambda \tilde{w}$. The proof of the remaining assertions is clear.

(2) Let $w'$ be a coarsening of $v'$. Then $U^1_w \subseteq U^1_v$, thus by Lemma 2 it follows that $K^Z_w \subseteq K^Z_v$. (This is actually true for any Galois extension $\tilde{K}|K$ and any $\tilde{w}$ coarsening of $\tilde{v}$, correspondingly.) Further, by general decomposition theory, $K^T_w|K^Z_w$ is the unique minimal one among all the sub-extensions of $K'|K^Z_w$ having residue field equal to $(Kw)'$. (Again, this is true for any Galois extension $\tilde{K}|K$, etc.) Now since by hypothesis $K^Z_w \subseteq K^Z_v \subseteq \Lambda$, we have: $K^T_w \subseteq \Lambda$, provided $\Lambda w' = (Kw)'$. Therefore, $w \in \mathcal{V}^0_{\Lambda,v} \iff K^T_w \subseteq \Lambda$, by the discussion above. □

**Definition 5.** By general valuation theory, the set $\mathcal{V}^0_{\Lambda,v}$ has an infimum whose valuation ring is the union of all the valuation rings $\mathcal{O}_w$ with $w \in \mathcal{V}^0_{\Lambda,v}$. We denote this valuation by

$$v_\Lambda := \inf \mathcal{V}^0_{\Lambda,v}$$

and call it the $\mathbb{Z}/\ell$**-abelian** $\Lambda$**-core** of $v$.

**Proposition 6.** *In the above context and notations, suppose that $\Lambda \neq K'$ is a proper sub-extension of $K'|K$ containing $K^Z$, thus in particular, $K^Z \neq K'$. Then the $\mathbb{Z}/\ell$-abelian $\Lambda$-core $v_\Lambda$ of $v$ is non-trivial and lies in $\mathcal{V}^0_{v,\Lambda}$. Consequently:*

*(1) If $v_1$ is a valuation of $K$ satisfying $v_1 < v_\Lambda$, then $\Lambda v'_1 \neq (Kv_1)'$. If $\Lambda v' = (Kv)'$, then $\Lambda v'_\Lambda = (Kv)'$, and $v_\Lambda$ is the minimal coarsening of $v$ with this property.*
*(2) $(Kv_Z)' = Kv_Z$ if and only if $(Kv)' = Kv$ if and only if $U_v/\ell = U^1_v/\ell$, where $v_Z$ is the $\mathbb{Z}/\ell$-abelian $K^Z$-core of $v$.*
*(3) If $v$ has rank one, or if $Kv \neq (Kv)'$, then $v$ equals its $\mathbb{Z}/\ell$-abelian $K^Z$-core $v_Z$.*

*Proof.* If $\mathcal{V}_{\Lambda,v}$ is empty, i.e., $\Lambda v' \neq (Kv)'$, then $\mathcal{V}^0_{\Lambda,v} = \{v\}$, hence $v_\Lambda = v$, and there is nothing to show. Now suppose that $\mathcal{V}_{\Lambda,v}$ is non-empty. Then $\Lambda v' = (Kv)'$, hence $v \in \mathcal{V}_{\Lambda,v}$, and we will show that actually $v_\Lambda \in \mathcal{V}_{\Lambda,v}$. Equivalently, by Lemma 4 (2), above, we have to show that $K^T_{v_\Lambda} \subseteq \Lambda$. Thus by the description of $K^T$ given in Lemma 2 (2), we have to show that for every given $v_\Lambda$-unit $x$, one has: $\sqrt[\ell]{x} \in \Lambda$. On the other hand, $v_\Lambda = \inf w$, $w \in \mathcal{V}_{\Lambda,v}$. Hence $\exists\, w \in \mathcal{V}_{\Lambda,v}$ such that $x$ is a $w$-unit. But then since $w \in \mathcal{V}_{\Lambda,v}$, by Lemma 4 (2), it follows that $K^T_w \subseteq \Lambda$. As $\sqrt[\ell]{x} \in K^T_w$, we finally get $\sqrt[\ell]{x} \in \Lambda$, as claimed.

The assertions (1), (2), (3) are immediate consequences of the main assertion of the Proposition proved above, and we omit their proof. □

**Proposition 7.** *(1) Let $v_1, v_2$ be valuations of $K$ such that $K^Z_{v_1}$, $K^Z_{v_2}$ are contained in some $\Lambda \neq K'$. Then the $\mathbb{Z}/\ell$-abelian $\Lambda$-cores of $v_1$ and $v_2$ are comparable.*

*(2) Let $v_1, v_2$ be valuations of $K$ which equal their $\mathbb{Z}/\ell$-abelian $K^Z$-cores, respectively. If $K^Z_{v_1} = K^Z_{v_2}$, then $v_1$ and $v_2$ are comparable.* (*Obviously, if $K^Z_w \subset K^Z_v$ strictly, then $w < v$ strictly.*)

*Proof.* (1) We first make the following observation: Suppose that $v_1$ and $v_2$ are independent valuations of $K$. Then $K^\times = U^1_{v_1} \cdot U^1_{v_2}$. (Indeed, this follows immediately from the Approximation Theorem for independent valuations). In particular, if $v_1$ and $v_2$ are independent, then $K'$ equals the compositum $K^Z_{v_1} K^Z_{v_2}$ inside $K'$. Now since by hypothesis we have $K^Z_{v_1}, K^Z_{v_2} \subseteq \Lambda \neq K'$, it follows that $v_1$ and $v_2$ are not independent. Let $v$ be the maximal common coarsening of $v_1$ and $v_2$. (By general valuation theory, the valuation ideal $\mathbf{m}_v$ is the maximal common ideal of $\mathcal{O}_{v_1}$ and $\mathcal{O}_{v_2}$.) Denote $w_i = v_i/v$ on the residue field $L := Kv$. Then we have:

- If both $w_1$ and $w_2$ are non-trivial, then they are independent.
- $L^Z_{w_i} = (K^Z_{v_i})v' \subseteq \Lambda v' \subseteq K'v' = (Kv)'$ for $i = 1, 2$.

Therefore, by the discussion above, we either have $\Lambda v' = (Kv)'$, or otherwise at least one of the $w_i$ is the trivial valuation.

First, we consider the case when one of the $w_i$ is trivial. Equivalently, we have $v_i = \min(v_1, v_2)$, hence $v_1$ and $v_2$ are comparable. Thus any two coarsenings of $v_1$ and $v_2$ are comparable, hence their $\mathbb{Z}/\ell$-abelian $\Lambda$-cores are comparable, too.

Second, consider the case where $\Lambda v' = (Kv)'$. Then by the definition of $\mathcal{V}'_{\Lambda, v_i}$, it follows that $v \in \mathcal{V}'_{\Lambda, v_i}$, as $v$ is by definition a coarsening of $v_i$, $i = 1, 2$. Hence finally the $\mathbb{Z}/\ell$-abelian $\Lambda$-cores of $v_1$ and $v_2$ are both some coarsenings of $v$, thus comparable.

(2) We apply assertion (1) with $\Lambda = K^Z_{v_1} = K^Z_{v_2}$. □

## 10.3 Hilbert decomposition in $\mathbb{Z}/\ell$ abelian-by-central extensions

We keep the notations from the introduction and the previous sections concerning field extensions $K|k$ and the canonical projection $pr : \overline{\Pi}^{\rm c}_K \to \overline{\Pi}_K$.

**Fact/Definition 8** *In the above notations we have the following:*

*(1) For a family $\Sigma = (\sigma_i)_i$ of elements of $\overline{\Pi}_K$, let $\Delta_\Sigma$ be the closed subgroup generated by $\Sigma$. Then the following are equivalent:*

*(i) there are preimages $\sigma'_i \in \overline{\Pi}^{\rm c}_K$, for all i, which commute with each other.*
*(ii) The preimage $\Delta''_\Sigma$ in $\overline{\Pi}^{\rm c}_K$ of $\Delta_\Sigma$ is abelian.*

*We say that a family of elements $\Sigma = (\sigma_i)_i$ of $\overline{\Pi}_K$ is* ***commuting liftable****, for short c.l., if $\Sigma$ satisfies the above equivalent conditions (i), (ii).*

*(2) For a family $(\Delta_i)_i$ of subgroups of $\overline{\Pi}_K$ the following are equivalent:*

*(i) All families $(\sigma_i)_i$ with $\sigma_i \in \Delta_i$ are c.l.*
*(ii) If $\Delta''_i$ is the preimage of $\Delta_i$ in $\overline{\Pi}^{\rm c}_K$, then $[\Delta''_i, \Delta''_j] = 1$ for all $i \neq j$.*

*We say that a family of subgroups $(\Delta_i)_i$ of $\overline{\Pi}_K$, is* ***commuting liftable****, for short c.l., if it satisfies the equivalent conditions (i), (ii) above.*

*(3) We will say that a subgroup $\Delta$ of $\overline{\Pi}_K$ is* ***commuting liftable****, for short c.l., if its preimage $\Delta''$ in $\overline{\Pi}^{\rm c}_K$ is commutative.*

*(4) We finally notice the following: For subgroups $T \subseteq Z$ of $\overline{\Pi}_K$, let $T'' \subseteq Z''$ be their preimages in $\overline{\Pi}^{\rm c}_K$. Then the following are equivalent:*

*(i) The pair $(T, Z)$ is c.l.*
*(ii) $T''$ is contained in the center of $Z''$.*

*(5) In particular, given a closed subgroup $Z$ of $\overline{\Pi}_K$, there exists a unique maximal (closed) subgroup $T$ of $Z$ such that $T$ and $(T, Z)$ are c.l. Indeed, denoting by $Z''$ the preimage of $Z$ in $\overline{\Pi}^{\rm c}_K$, and denoting by $T''$ its center, the group $T$ is the image of $T''$ in $\overline{\Pi}_K$ under the canonical projection $pr : \overline{\Pi}^{\rm c}_K \to \overline{\Pi}_K$.*

We next recall the following fundamental fact concerning $\mathbb{Z}/\ell$ **liftable commuting pairs**. It might be well possible that one could work out a proof along the technical steps in the proofs from Bogomolov–Tschinkel [BT02]. But there is a much simpler/easier way to get the result by using the theory of **rigid elements**, originating in work by Ware [War81], and further developed by Arason–Jacob–Ware [AJW86], Koenigsmann [Ko01], and others; see Topaz [ToXX] for comprehensive proofs, in particular for the Galois translation of the theory of rigid elements into that of commuting liftable pairs.

**Key Fact 9** *Let $K|k$ be an extension with $k$ algebraically closed of characteristic $\neq \ell$. In the notations from above, let $\sigma, \tau \in \overline{\Pi}_K$ be c.l. elements of $\overline{\Pi}_K$ such that the closed subgroup $<\sigma, \tau>$ generated by $\sigma, \tau$ is not pro-cyclic. Then there exists a valuation $v$ of $K$ with the following properties:*

*(i) The group $<\sigma, \tau>$ is contained in $Z_v$.*
*(ii) The intersection $<\sigma, \tau> \cap T_v$ is non-trivial, and* $\mathrm{char}(Kv) \neq \ell$.

*Proof.* (Sketch, see Topaz [ToXX] for details.) Let $T \subset K^\times/\ell$ be the orthogonal complement of $<\sigma, \tau>$ under the Kummer pairing. Equivalently, $K_T := K[\sqrt[\ell]{T}\,]$ is the fixed field of $<\sigma, \tau>$ in $K'$, and $(K^\times/\ell)/T$ is the Kummer dual of $<\sigma, \tau>$ under the Kummer pairing. Let $x, y \in K^\times$ be such that $K^\times/\ell$ is generated by $T, x, y$ as an abelian group. Then the fact that $\sigma, \tau$ is a commuting liftable pair implies that if $\chi_x, \chi_y \in \mathrm{Hom}(\overline{\Pi}_K, \mu_\ell)$ are the characters defined by $x, y$, it follows that their cup product $\chi_x \cup \chi_y \in \mathrm{H}^2(K, \mu_\ell^{\otimes 2})$ is non-trivial. From this fact it follows instantly that $T + Tz \subset \cup_{1=0}^{\ell-1} z^i T$ for all $z \in K^\times/\ell$. Thus all the elements of $K^\times/\ell$ are quasi rigid with respect to $T$. One concludes that there $T$-rigid elements in $K^\times$, thus there exists a valuation $v$ of $K$ such that $1 + \mathbf{m}_v \subseteq T$, and $vK$ is not divisible by $\ell$, and $\mathrm{char}(Kv) \neq \ell$, etc. □

### 10.3.1 Inertia elements

Recall that in the notations from above, we say that an element $\sigma \in \overline{\Pi}_K$ is an $\ell$-**inertia element**, for short **inertia element**, if there exists a valuation $v$ of $K$ such that $\sigma \in T_v$ and $\mathrm{char}(Kv) \neq \ell$. Clearly, the set of all the inertia elements at $v$ is exactly $T_v$.

**Lemma 10.** *Let $\sigma \neq 1$ be an inertia element of $\overline{\Pi}_K$. Then there exists a valuation $v_\sigma$ of $K$, which we call the canonical valuation for $\sigma$ such that the following hold:*

*(i)* $\sigma \in T_{v_\sigma}$, *i.e., $\sigma$ is inertia element at $v_\sigma$.*
*(ii)* *If $\sigma$ is inertia element at some valuation $v$, then $v_\sigma \leq v$.*

*Proof.* We construct $v_\sigma$ as follows. Let $\Lambda$ be the fixed field of $\sigma$ in $K'$. For every valuation $v$ such that $\sigma \in T_v$, let $v_\Lambda$ be the $\mathbb{Z}/\ell$-abelian $\Lambda$-core of $v$. We claim that $v_\sigma := v_\Lambda$ satisfies the conditions (i) and (ii). Indeed, since $\sigma \in T_v$, and $K^T v' = (Kv)'$, we have $K^T_v \subseteq \Lambda$, hence $\Lambda v' = (Kv)'$. Thus by Proposition 6 (1), it follows that $\Lambda v'_\Lambda = (K v_\Lambda)'$. But then one must have $K^T_{v_\Lambda} \subseteq \Lambda$, and therefore $\mathrm{Gal}(K'|\Lambda) \subseteq T_{v_\Lambda}$. Hence $\sigma \in T_{v_\Lambda}$ verifies (i).

In order to prove (ii), let $v_1$ be another valuation of $K$ such that $\sigma \in T_{v_1}$. For the $\mathbb{Z}/\ell$-abelian $\Lambda$-core $v_{1,\Lambda}$ of $v_1$, by the discussion above, we have $\Lambda v'_{1,\Lambda} = (K v_{1,\Lambda})'$. We claim that actually $v_\Lambda = v_{1,\Lambda}$. Indeed, both $v_\Lambda$ and $v_{1,\Lambda}$ equal their $\mathbb{Z}/\ell$-abelian $\Lambda$-cores. Hence they are comparable by Proposition 7 (1). By contradiction, suppose that $v_{1,\Lambda} \neq v_\Lambda$, say $v_{1,\Lambda} < v_\Lambda$. Since $v_\Lambda$ equals its $\mathbb{Z}/\ell$-abelian $\Lambda$-core, and $v_{1,\Lambda} < v_\Lambda$, it follows by Proposition 6 (1), that $\Lambda v'_{1,\Lambda} \neq (K v_{1,\Lambda})'$, contradiction. Thus $v_\Lambda = v_{1,\Lambda}$. Since $v_{1,\Lambda} \leq v_1$, we finally get $v_\Lambda \leq v_1$. This completes the proof of (ii). □

**Proposition 11.** *In the context and the notations from above, the following hold:*

*(1) Let $\Sigma = (\sigma_i)_i$ be a c.l. family of inertia elements. Then the canonical valuations $v_{\sigma_i}$ are pairwise comparable. Moreover, denoting by $v_\Sigma = \sup_i v_{\sigma_i}$ their supremum, and by $\Lambda$ the fixed field of $\Sigma$ in $K'$, one has:*

*(a)* $\sigma_i \in T_{v_\Sigma}$ *for all $i$.*
*(b)* $v_\Sigma$ *equals its $\mathbb{Z}/\ell$-abelian $\Lambda$-core.*

*(2) Let $Z \subseteq \overline{\Pi}_K$ be some subgroup, and $\Sigma_Z = (\sigma_i)_i$ be the family of all inertia elements $\sigma_i$ in $Z$ such that $(\sigma_i, Z)$ is c.l. for each $i$. Then the valuation $v := v_{\Sigma_Z}$ as constructed above with respect to $\Sigma_Z$ satisfies:*

*(a)* $Z \subseteq Z_v$.
*(b)* $\Sigma_Z = Z \cap T_v$.

*Proof.* (1) We may assume that all $\sigma_i$ are nontrivial. For each $\sigma_i$ let $T_i$ be the closed subgroup of $\overline{\Pi}_K$ generated by $\sigma_i$, and $\Lambda_i$ the fixed field of $\sigma_i$ in $K'$. Thus $T_i = \mathrm{Gal}(K'|\Lambda_i)$, and $\sigma_i \neq 1$ implies that $T_i \cong \mathbb{Z}/\ell$. Setting $T := T_i \cap T_j$, we have the following possibilities:

Case $T \neq \{1\}$: Since $T_i \cong \mathbb{Z}/\ell \cong T_j$, we have $T = T_i = T_j$, and $\Lambda = \Lambda_i = \Lambda_j$ is the fixed field of $T$ in $K'$. Hence by Proposition 7 (2), the $\mathbb{Z}/\ell$-abelian $\Lambda$-core of $v_{\sigma_i}$ and $v_{\sigma_j}$ are comparable. But then reasoning as at the end of the proof of (ii) from Lemma 10, it follows that $v_{\Lambda,i} = v_{\Lambda,j}$. Hence finally $v_{\sigma_j} = v_{\sigma_i}$, as claimed.

Case $T = \{1\}$: Let $T_{ij}$ be the subgroup $T_{ij}$ generated by $\sigma_i, \sigma_j$ in $\overline{\Pi}_K$, and let $\Lambda_{ij}$ be the fixed field of $T_{ij}$ in $K'$. Since $T_i \cap T_j = \{1\}$, we have $T_{ij} \cong \mathbb{Z}/\ell \times \mathbb{Z}/\ell$, thus $T_{ij}$ is not pro-cyclic. Hence by the Key Fact 9 above, there exists a valuation $v$ such that $T_{ij} \subseteq Z_v$, $T := T_v \cap T_{ij}$ is non-trivial, and $\mathrm{char}(Kv) \neq \ell$. Moreover, by replacing $v$ by its $\mathbb{Z}/\ell$-abelian $\Lambda_{ij}$-core, we can suppose that actually $v$ equals its $\Lambda_{ij}$-core.

Finally let us notice that we have $T_{ij}/T = \mathrm{Gal}\big((Kv)'|\Lambda_{ij}v'\big)$, and by Kummer theory, $\mathrm{Gal}\big((Kv)'|\Lambda_{ij}v'\big)$ is of the form $(\mathbb{Z}/\ell)^r$ for some $r$. Now since $T$ is non-trivial, and $T_{ij} \cong \mathbb{Z}/\ell \times \mathbb{Z}/\ell$, we finally get: $T_{ij}/T$ is either trivial, or $T_{ij}/T \cong \mathbb{Z}/\ell$ else.

Now let $v_i$ be the $\mathbb{Z}/\ell$-abelian $\Lambda_i$-core of $v$. The we have the following case discussion:

Suppose $v_i > v_{\sigma_i}$: Reasoning as in the proof of (ii) of Lemma 10, by Proposition 6 (1), we get: Since $v_i$ is the $\mathbb{Z}/\ell$-abelian $\Lambda_i$-core of $v$, and $v_{\sigma_i} < v_i$, we have $\Lambda_i v'_{\sigma_i} \neq (Kv_{\sigma_i})'$, contradiction. The same holds correspondingly for $v_{\sigma_j}$ and the corresponding $v_j$. Hence we must have $v_i \leq v_{\sigma_i}$, and $v_j \leq v_{\sigma_j}$.

Suppose $v_i < v_{\sigma_i}$: Recall that $v_{\sigma_i}$ equals its $\mathbb{Z}/\ell$-abelian $\Lambda_i$-core, and further $\Lambda_i v'_{\sigma_i} = (Kv_{\sigma_i})'$ by the definition/construction of $v_{\sigma_i}$. Since $v_i < v_{\sigma_i}$, it follows by Proposition 6, (1), that $\Lambda_i v'_i \neq (Kv_i)'$. But then by Proposition 6, (3), it follows that $v_i = v$. Hence finally we have the following situation: $v = v_i < v_{\sigma_i}$, and $\Lambda_i v' \neq (Kv)'$. Equivalently, $\Lambda_i \not\subseteq K_v^T$, and so, $T_i \not\subseteq T_v$. On the other hand, since $\overline{\Pi}_K$ is $\ell$-torsion, it follows that $T_i \cap T_v$ is trivial, hence $T_i \cap T$ is trivial. Hence by the remarks above we have: $T_{ij} = T_i T$. On the other hand, $v < v_{\sigma_i}$ implies that $T_v \subseteq T_{v_{\sigma_i}}$, and therefore $T \subseteq T_{v_{\sigma_i}}$. Since $T_i \subseteq T_{v_{\sigma_i}}$ by the definition of $v_{\sigma_i}$, we finally get: $T_{ij}$ is contained in $T_{v_{\sigma_i}}$. Therefore, $\sigma_i, \sigma_j$ are both inertia elements at $v_{\sigma_i}$. But then reasoning as at the end of the proof of (ii) from Lemma 10, we deduce that the $\mathbb{Z}/\ell$-abelian $\Lambda_j$-core of $v_{\sigma_i}$, say $w_i$, equals $v_{\sigma_j}$. Thus finally we have $v_{\sigma_j} = w_i \leq v_{\sigma_i}$, hence $v_{\sigma_j}$ and $v_{\sigma_i}$ are comparable, as claimed.

By symmetry, we come to the same conclusion in the case $v_j < v_{\sigma_j}$, etc. Thus it remains to analyze the case when $v_{\sigma_i} = v_i$ and $v_{\sigma_j} = v_j$. Now since both $v_i$ and $v_j$ are coarsenings of $v$, it follows that they are comparable. Equivalently, $v_{\sigma_i} = v_i$ and $v_{\sigma_j} = v_j$ are comparable, as claimed.

(2) Let Let $\sigma \in \Sigma_Z$ be a non-trivial element, and let $v_\sigma$ be the canonical valuation attached to $\sigma$ as defined above at Lemma 10.

*Claim.* $Z \subseteq Z_{v_\sigma}$.

Case $Z$ is pro-cyclic:Then $Z \cong \mathbb{Z}/\ell$, and if $T_\sigma$ is the subgroup of $Z$ generated by $\sigma$, then $T_\sigma = Z$. Hence $Z$ consists of inertia elements at $v_\sigma$ only, and in particular, $Z \subseteq Z_{v_\sigma}$.

Case $Z$ is not pro-cyclic: Let $\tau \in Z$ be any element such that the closed subgroup $Z_{\sigma,\tau}$ generated by $\sigma, \tau$ is not pro-cyclic. We claim that $\tau \in Z_{v_\sigma}$. Indeed, by the Key Fact 9 above, it follows that there exists a valuation $v$ having the following properties: $Z_{\sigma,\tau} \subseteq Z_v$, $T := Z_{\sigma,\tau} \cap T_v$ is non-trivial, etc. Let $\rho$ be a generator of $T$. Then since $(\sigma, Z)$ is c.l., and $\rho \in Z$, it follows that $(\sigma, \rho)$ is a c.l. pair of inertia elements of $\overline{\Pi}_K$. Hence by assertion (1) above, it follows that the canonical valuations $v_\sigma$ and $v_\rho$ are comparable. We notice that $v_\rho \leq v$, as the former valuation is a core of the latter one. We have the following case by case discussion:

- Suppose that $v_\sigma \leq v_\rho$: Then $v_\sigma \leq v_\rho \leq v$, hence $Z_v \subseteq Z_{v_\rho} \subseteq Z_{v_\sigma}$. Now since $\tau \in Z_{\sigma,\tau} \subseteq Z_v$, we finally get $\tau \in Z_{v_\sigma}$, as claimed.

- Suppose that $v_\sigma > v_\rho$: Then in the notations from above, $\Lambda_\sigma v'_\rho \neq (Kv_\rho)'$, hence $T_\sigma$ is mapped isomorphically into the residual Galois group $\mathrm{Gal}(K'v'_\rho|\Lambda_\sigma v'_\rho)$. In particular, since $\rho \in T_{v_\rho}$, it follows that $T_\sigma \cap T_\rho$ is trivial, hence $\sigma, \rho$ generate

$Z_{\sigma,\tau}$. On the other hand, $v_\sigma > v_\rho$ implies $T_{v_\rho} \subseteq T_{v_\sigma} \subseteq Z_{v_\sigma}$. Since $\rho \in T_{v_\rho}$, we finally get $\rho \in Z_{v_\sigma}$. Hence finally $\tau \in Z_{v_\sigma}$.

Combining both cases conclude the proof of the claim.

Now recall that by (1), the valuations $v_\sigma$ are comparable, and that $v$ denotes their supremum. By general decomposition theory for valuation, since $Z \subseteq Z_{v_\sigma}$ for all $\sigma$, one has $Z \subseteq Z_v$ too. In order to show that $\Sigma_Z = Z \cap T_v$, we notice the following. First, $\Sigma_Z \subseteq T_v$ by the assertion (1) above. For the converse, recall that for every valuation $v$ of $K$, the decomposition groups $Z_v(\ell)$ at $v$ in the maximal pro-$\ell$ quotient $G_K(\ell)$ of $G_K$ have the following structure, see e.g. Pop [Pop06],

$$Z_v(\ell) = T_v(\ell) \cdot G,$$

where $G$ is a complement of $T_v(\ell)$, thus isomorphic to $G_{Kv}(\ell)$. Moreover, $T_v(\ell)$ is abelian, and $G$ and $T_v(\ell)$ commute element-wise with each other. Thus each element $\sigma \in T_v \subseteq Z_v \subseteq \overline{\Pi}_K$ has a preimage in $Z_v(\ell)$ which lies in the center of $Z_v(\ell)$. But then its projection to $\overline{\Pi}_K^{\mathrm{c}}$ will commute with the preimage of $Z_v$ in $\overline{\Pi}_K^{\mathrm{c}}$. Coming back to the proof of assertion (2) we finally have: If $\sigma \in Z \cap T_v$, then $(\sigma, Z)$ is c.l. Hence $\sigma \in \Sigma_Z$ by the definition of $\Sigma_Z$. □

### *10.3.2 Inertia elements and the c.l. property*

Proposition 11 has the following consequence.

**Proposition 12.** *(1) Let $\Delta$ be a c.l. subgroup of $\overline{\Pi}_K$. Then $\Delta$ contains a subgroup $\Sigma$ consisting of inertia elements such that $\Delta/\Sigma$ is pro-cyclic (maybe trivial). In particular, there exists a valuation $v := v_\Sigma$ such that $\Delta \subseteq Z_v$, and $\Delta \cap T_v = \Sigma$, and $v$ equals its $\mathbb{Z}/\ell$-abelian $\Lambda$-core, where $\Lambda$ is the fixed field of $\Sigma$ in $K'$.*

*(2) Let $Z \subseteq \overline{\Pi}_K$ be a closed subgroup, and $\Sigma_Z$ a maximal subgroup of $Z$ such that the following are satisfied:*

*($\star$) $(\Sigma_Z, Z)$ is c.l.*

*Suppose that $\Sigma_Z \neq Z$. Then $\Sigma_Z$ is the unique maximal subgroup of $Z$ satisfying $(\star)$, and it consists of all the inertia elements $\sigma$ in $Z$ such that $(\sigma, Z)$ is c.l. Moreover, there exists a unique valuation $v$ such that $Z \subseteq Z_v$, $\Sigma_Z = Z \cap T_v$, and $v$ equals its $\mathbb{Z}/\ell$-abelian $\Lambda$-core, where $\Lambda$ is the fixed field of $\Sigma_Z$ in $K'$.*

*Proof.* (1) When all $\sigma \in \Delta$ are inertia elements, then $\Sigma = \Delta$ and the conclusion follows by applying Proposition 11 (1).

Now we assume that there is an element $\sigma_0 \in \Delta$ which is not an inertia element. Then for each $\sigma_i' \in \Delta$ such that the closed subgroup $Z_{\sigma_0,\sigma_i'}$ generated by $\sigma_0, \sigma_i'$ is not pro-cyclic, the following holds. Since $(\sigma_i', \sigma_0)$ is by hypothesis c.l., it follows that there exists a valuation $v_i$ of $K$ such that $Z_{\sigma_i',\sigma_0} \subseteq Z_{v_i}$, and $T_i := Z_{\sigma_i',\sigma_0} \cap T_{v_i}$ is not trivial. Moreover, since $\sigma_0$ is by assumption not an inertia element, it follows that denoting by $\sigma_i$ a generator of $T_i$, we have: $\sigma_0, \sigma_i$ generate topologically $Z_{\sigma_i',\sigma_0}$, and

$\sigma_i$ is an inertia element in $\Delta$. In particular, if $v_{\sigma_i}$ is the canonical valuation attached to the inertia element $\sigma_i$, then $v_{\sigma_i} \le v_i$. Hence we have $\sigma_0 \in Z_{v_i} \subseteq Z_{v_{\sigma_i}}$. We next apply Proposition 11 and get the valuation $v := v_\Sigma = \sup_i v_{\sigma_i}$. Since $\sigma_0 \in Z_{v_{\sigma_i}}$ for all $i$, by general valuation theory one has $\sigma_0 \in Z_{v_\Sigma}$.

(2) We first claim that $\Sigma_Z$ consists of inertia elements. By contradiction, suppose that this is not the case, and let $\sigma_0 \in \Sigma_Z$ be a non-inertia element. Reasoning as in (1) above and in the above notations we have: For every $\sigma'_i \in Z$ such that the closed subgroup $Z_{\sigma'_i,\sigma_0}$ generated by $\sigma_0, \sigma'_i$ is not pro-cyclic, there exists an inertia element $\sigma_i \in Z_{\sigma'_i,\sigma_0}$, such that $\sigma'_i \in Z_{v_{\sigma_i}}$, and the closed subgroup generated by $\sigma_i, \sigma_0$ equals $Z_{\sigma'_i,\sigma_0}$. Then if $v_{\sigma_i}$ is the canonical valuation for $\sigma_i$, we have $Z_{\sigma'_i,\sigma_0} \subseteq Z_{v_{\sigma_i}}$, and $Z_{\sigma'_i,\sigma_0} \cap T_{v_{\sigma_i}}$ is generated by $\sigma_i$. In particular, if $\Lambda_0$ is the fixed field of $\sigma_0$ in $K'$, then for every $\sigma_i$ one has $\Lambda_0 v'_{\sigma_i} \ne (K v_{\sigma_i})'$, hence $v_{\sigma_i}$ equals its $\mathbb{Z}/\ell$-abelian $\Lambda_0$-core. Now taking into account that for all subscripts $i$ one has $K^Z_{v_{\sigma_i}} \subset \Lambda_0$, it follows by Proposition 7, (1), that $(v_{\sigma_i})_i$ is a family of pairwise comparable valuations. Let $v = \sup_i v_{\sigma_i}$ be the supremum of all these valuations. Since $v_{\sigma_i} \le v$ for all subscripts $i$, it follows that $T_{v_{\sigma_i}} \subseteq T_v$ for all $\sigma_i$; and since we supposed that $\sigma_0$ is not an inertia element, one has $\sigma_0 \notin T_v$. On the other hand, the set of all the $\sigma_i$ together with $\sigma_0$ generate $Z$; hence $T := Z \cap T_v$ together with $\sigma_0$ topologically generates $Z$. But then $(\sigma_0, T)$ is c.l., and therefore $Z = \langle \sigma_0, T \rangle$ is c.l., contradiction!

Hence $\Sigma_Z$ consists of inertia elements only, and by hypothesis, $\Sigma_Z$ is c.l. We next let $v := v_{\Sigma_Z}$ be the valuation constructed in Proposition 11 (1), for the c.l. family consisting of all the elements from $\Sigma_Z$.

*Claim.* $Z \subseteq Z_v$.

Indeed, let $\sigma_0 \in Z$ be a fixed element $\ne 1$, and $\sigma'_i \in \Sigma_Z$. Then $(\sigma'_i, \sigma_0)$ is c.l. by hypothesis of the Proposition. Reasoning as in (1) above, we obtain $\sigma_i$ and $v_{\sigma_i}$ as there. On the other hand, $\sigma'_i \in \Sigma_Z$ is itself an inertia element, as $\Sigma_Z$ consists of inertia elements only by the discussion above. Since $(\sigma'_i, Z)$ is c.l. by hypothesis, it follows that $(\sigma'_i, \sigma_i)$ is c.l. Since they are inertia elements, it follows by Proposition 11 (1), that $v_{\sigma_i}$ and $v_{\sigma'_i}$ are comparable. And further note that $\sigma_0 \in Z_{v_{\sigma_i}}$. We have the following case by case discussion.

- If $v_{\sigma_i} \ge v_{\sigma'_i}$ for all $i$, then setting $v_0 := v_{\sigma_i}$ for a fixed $i$, we have

$$v_0 \ge \sup_i v_{\sigma'_i} = v_{\Sigma_Z} = v.$$

Hence $Z_{v_0} \subseteq Z_v$. Since $\sigma_0 \in Z_{v_0}$, we finally find $\sigma_0 \in Z_v$, as claimed.

- If $v_{\sigma_i} < v_{\sigma'_i}$ for some $\sigma'_i$, then $\sigma'_i \notin T_{v_{\sigma_i}}$, hence $(\sigma_i, \sigma'_i)$ and $(\sigma_0, \sigma'_i)$ generate the same closed subgroup. Or equivalently, $\sigma_0$ is contained in the subgroup generated by $(\sigma_i, \sigma'_i)$. On the other hand, $v_{\sigma_i} < v_{\sigma'_i} \le v$ implies $T_{v_{\sigma_i}} \subseteq T_{v_{\sigma'_i}} \subseteq T_v$, hence $\sigma_i, \sigma'_i \in T_v$. But then $\sigma_0 \in T_v$, and in particular, $\sigma_0 \in Z_v$, as claimed.

Finally we prove that $\Sigma_Z = Z \cap T_v$. By the definition of $v$ we have $\Sigma_Z \subseteq T_v$, hence $\Sigma_Z \subseteq Z \cap T_v$. For the converse, we apply Proposition 11 (2). □

## 10.4 Almost quasi $r$-divisorial subgroups

In this section we will prove a more general form of the main result announced in the Introduction. We begin by quickly recalling some basic facts about (transcendence) defectless valuations, see e.g. Pop [Pop06] Appendix for more details.

### *10.4.1 Generalized quasi divisorial valuations*

Let $K|k$ be a function field over the algebraically closed field $k$ with $\mathrm{char}(k) \neq \ell$. For every valuation $v$ on $K$ (and/or on any algebraic extension of $K$, like for instance $K'$ or $K''$), since $k$ is algebraically closed, the group $vk$ is a totally ordered $\mathbb{Q}$-vector space (which is trivial, if the restriction of $v$ to $k$ is trivial). We will denote by $r_v$ the rational rank of the torsion free group $vK/vk$, and by abuse of language call it the **rational rank of** $v$. Next notice that the residue field $kv$ is algebraically closed too, and $Kv\,|kv$ is some field extension, but not necessarily a function field. We will denote $\mathrm{td}_v = \mathrm{td}(Kv\,|\,kv)$ and call it the **residual transcendence degree**. By general valuation theory, see e.g. [BOU] Ch.6 §10.3,

$$r_v + \mathrm{td}_v \leq \mathrm{td}(K|k)$$

and we will say that $v$ has **no (transcendence) defect**, or that $v$ is **(transcendence) defectless**, if the above inequality is an equality.

Using Fact 5.4 from [Pop06], it follows that for a valuation $v$ of $K$ and $r \leq \mathrm{td}(K|k)$ the following are equivalent:

(i) $v$ is minimal among the valuations $w$ satisfying $r_w = r$ and $\mathrm{td}_w = \mathrm{td}(K|k) - r$.

(ii) $v$ has no relative defect and satisfies $r_v = r$ and $r_{v'} < r$ for any proper coarsening $v'$ of $v$.

**Definition 13.** A valuation of $K$ with the equivalent properties (i) and (ii), above is called **almost quasi $r$-divisorial**, or an **almost quasi $r$-divisor** of $K|k$, or simply a **generalized almost quasi divisor**, if the rank $r$ is not relevant for the context.

*Remark 14.* (1) The additivity of the rational rank $r_{(\cdot)}$, see [Pop06] Fact 5.4 (1), implies, if $v$ is almost quasi $r$-divisorial on $K|k$, and $v_0$ is almost quasi $r_0$-divisorial on $Kv\,|\,kv$, then the compositum $v_0 \circ v$ is an almost quasi $(r + r_0)$-divisor of $K$.

(2) Similar to [Pop06] Appendix, Fact 5.5 (2) (b), if $v$ is almost quasi $r$-divisorial, then $Kv|kv$ is a function field with $\mathrm{td}(Kv\,|\,kv) = \mathrm{td}(K|k) - r$, and $vK/vk \cong \mathbb{Z}^r$.

(3) A prime divisor of $K|k$ is an almost quasi 1-divisor. And conversely, an almost quasi 1-divisor $v$ of $K$ is a prime divisor if and only if $v$ is trivial on $k$.

**Proposition 15.** *In the above context, suppose that $d = \mathrm{td}(K|k) > 0$. Let $v$ be an almost quasi $r$-divisor of $K|k$, and $v''$ a prolongation of $v$ to $K''$. Let $T_{v''} \subseteq Z_{v''}$ be the inertia group, respectively decomposition group, of $v''$ in $\overline{\Pi}^{\mathrm{c}}_K$, and $G_{v''} = Z_{v''}/T_{v''}$ the*

*Galois group of the corresponding Galois residue field extension $K''v''\,|\,Kv$. Then the following hold:*

*(1)* $T_{v''} \cong (\mathbb{Z}/\ell)^r$*, and the canonical exact sequence* $1 \to T_{v''} \hookrightarrow Z_{v''} \to G_{v''} \to 1$ *is split, i.e.,* $T_{v''}$ *has complements in* $Z_{v''}$*. And* $Z_{v''} \cong T_{v''} \times G_{v''}$ *as profinite groups.*

*(2) Under* $pr : \overline{\Pi}^{\mathrm{c}}_K \to \overline{\Pi}_K$*, the exact sequence* $1 \to T_{v''} \hookrightarrow Z_{v''} \to G_{v''} \to 1$ *maps onto* $1 \to T_v \hookrightarrow Z_v \to \overline{\Pi}_{Kv} \to 1$*. Hence* $T_{v''}$ *maps onto* $T_v$*, and* $G_{v''}$ *maps onto* $\overline{\Pi}_{Kv}$*. In particular,* $(T_v, Z_v)$ *is c.l.*

*(3) Moreover, the following hold:*

*(a)* $Z_v$ *contains c.l. subgroups* $\cong (\mathbb{Z}/\ell)^d$ *and* $Z_v$ *is maximal among the subgroups* $Z'$ *of* $\overline{\Pi}_K$ *which contain subgroups* $T' \cong (\mathbb{Z}/\ell)^r$ *with* $(T', Z')$ *c.l.*

*(b)* $T := T_v$ *is the unique maximal subgroup of* $Z_v$ *with* $(T, Z_v)$ *c.l. Equivalently, if* $T''_v \subseteq Z''_v$ *are the preimages of* $T_v \subseteq Z_v$ *in* $\overline{\Pi}^{\mathrm{c}}_K$*, then* $T''_v$ *is the center of* $Z''_v$*.*

*Proof.* Assertions (1) and (2) follow immediately from the behavior of the decomposition/inertia groups in towers of algebraic extensions, and Fact 2.1 (2) (3) of [Pop06].

(3a) First let us prove that $Z_v$ contains c.l. subgroups $\Delta \cong (\mathbb{Z}/\ell)^d$. Since $v$ is an almost quasi $r$-divisor, we have $\mathrm{td}(Kv\,|\,kv) = \mathrm{td}_v = \mathrm{td}(K|k) - r$. Now if $\mathrm{td}_v = 0$, then we are done by assertion (1) above. If $\mathrm{td}_v > 0$, then we consider any almost quasi $\mathrm{td}_v$-divisor $v_0$ on the residue field $Kv$. Then denoting by $v_1 := v_0 \circ v$ the refinement of $v$ by $v_0$, we have $v < v_1$, hence $T_v \subseteq T_{v_1}$. And $d = \mathrm{td}(K|k) = r + \mathrm{td}_v = r_{v_1}$, hence $v_1$ is an almost quasi $d$-divisor of $K$. Thus $T_v$ is contained in the group $T_{v_1} \cong (\mathbb{Z}/\ell)^d$, and by assertion (2) above, it follows that $T_{v_1}$ is c.l.

Let $T \subseteq Z$ be a closed subgroup as in (3a) such that $Z_v \subseteq Z$. We claim that $Z = Z_v$. Let $\Sigma$ be a maximal c.l. subgroup of $Z$ such that $T \subseteq \Sigma$, and $(\Sigma, Z)$ is c.l. too. Applying Proposition 12, let $w := v_\Sigma$ be the resulting valuation from loc.cit. Hence we have $Z \subseteq Z_w$, and $\Sigma = Z \cap T_w$.

*Claim.* $w \geq v$.

Indeed, suppose by contradiction that $w < v$. Then by the fact that $v$ is an almost quasi $r$-divisor, it follows that $r_w < r = r_v$. But then

$$\dim(wK/\ell) = r_w < r = \dim(vK/\ell),$$

hence $T_w \cong (\mathbb{Z}/\ell)^{r_w}$. Since $T \cong (\mathbb{Z}/\ell)^r$ and $T \subseteq \Sigma \cong (\mathbb{Z}/\ell)^{r_w}$, we get a contradiction and the claim is proved. Therefore $Z_w \subseteq Z_v$, and since $Z \subseteq Z_w$, we finally have $Z \subseteq Z_v$, as claimed in assertion (3a).

(3b) By assertion (2) above, it follows that $T_v$ and $(T_v, Z_v)$ are c.l. We show that $T_v$ is the unique maximal (closed) subgroup of $Z_v$ with this property. Indeed, let $T$ be a closed subgroup of $Z_v$ as in (3a). Then denoting by $\Sigma$ the closed subgroup of $Z_v$ generated by $T_v$ and $T$, since $T_v$ and $T$, and $(T_v, Z_v)$ and $(T, Z_v)$ are c.l., it follows that $\Sigma$ and $(\Sigma, Z_v)$ are c.l. too. Thus w.l.o.g. we may assume that $T_v \subseteq T$, and that $T$ is maximal with the properties from (3b). Now if $r = d$, then $Z_v = T_v$, and there is nothing to prove. Thus suppose that $r < d$. Let $v_\Sigma$ be the unique valuation of $K$ given by Proposition 12.

*Claim.* $v_\Sigma \le v$.

Indeed, suppose by contradiction that $v_\Sigma > v$. Since $v$ is a quasi $r$-divisor of $K$, it follows that $Kv|kv$ is a function field with $\mathrm{td}(Kv|kv) = \mathrm{td}_v = d - r > 0$. Since the valuation $v_0 := v_\Sigma/v$ on $Kv$ is non-trivial, it follows that $Z_{v_0} \subsetneq (Kv)'$ is a proper subgroup of $\overline{\Pi}_K$. Taking into account that $Z_{v_\Sigma}$ is the preimage of $Z_{v_0} \subsetneq (Kv)'$ under the canonical projection $Z_v \to \mathrm{Gal}\big((Kv)'|Kv\big)$, it follows that $Z_{v_\Sigma}$ is strictly contained in $Z_v$, contradiction.

By the claim, we have $v_\Sigma \le v$ and so $T_{v_\Sigma} \subseteq T_v$, hence $\Sigma = Z_v \cap T_{v_\Sigma} \subseteq T_v$ and thus finally $T \subseteq T_v$, as claimed. □

### 10.4.2 Characterizing almost quasi $r$-divisorial subgroups

We keep the notations from the previous subsection.

**Definition 16.** We say that a closed subgroup $Z$ of $\overline{\Pi}_K$ is an **almost quasi $r$-divisorial subgroup**, if there exists an almost quasi $r$-divisor $v$ of $K|k$ such that $Z = Z_v$.

Below we give a characterization of the almost quasi $r$-divisorial subgroups of $\overline{\Pi}_K$, thus of the almost quasi $r$-divisors of $K$, in terms of the group theoretical information encoded in the Galois group $\overline{\Pi}{}^{\mathrm{c}}_K$ alone, provided $r < \mathrm{td}(K|k)$.

**Theorem 17.** *Let $K|k$ be a function field over the algebraically closed field $k$ with $\mathrm{char}(k) \neq \ell$. Let $pr : \overline{\Pi}{}^{\mathrm{c}}_K \to \overline{\Pi}_K$ be the canonical projection, and for subgroups $T$, $Z$, $\Delta$ of $\overline{\Pi}_K$, let $T''$, $Z''$, $\Delta''$ denote their preimages in $\overline{\Pi}{}^{\mathrm{c}}_K$. Then the following hold:*

*(1) The transcendence degree $d = \mathrm{td}(K|k)$ is the maximal integer $d$ such that there exists a closed subgroup $\Delta \cong (\mathbb{Z}/\ell)^d$ of $\overline{\Pi}_K$ with $\Delta''$ abelian.*

*(2) Suppose that $d := \mathrm{td}(K|k) > r > 0$. Let $T \subseteq Z$ be closed subgroups of $\overline{\Pi}_K$. Then $Z$ endowed with $T$ is an almost quasi $r$-divisorial subgroup of $\overline{\Pi}_K$ if and only if $Z$ is maximal in the set of closed subgroups of $\overline{\Pi}_K$ which satisfy:*

*(i) $Z$ contains a closed subgroup $\Delta \cong (\mathbb{Z}/\ell)^d$ such that $\Delta''$ is Abelian.*

*(ii) $T \cong (\mathbb{Z}/\ell)^r$, and $T''$ is the center of $Z''$.*

*Proof.* Recall from Fact/Definition 8, especially the points (3)–(5), that $\Delta''$ being abelian is equivalent to $\Delta$ being c.l., and $T''$ being the center of $Z''$ is equivalent to $T$ being the maximal subgroup of $Z$ such that $(T,Z)$ is c.l. We will use the c.l. terminology from now on.

(1) Let $\Delta$ be any c.l. closed non-procyclic subgroup of $\overline{\Pi}_K$. Then by Proposition 12 (1), it follows that there exists a valuation $v$ of $K$ such that $\Delta \subseteq Z_v$, and setting $T_\Delta := \Delta \cap T_v$, it follows that $\Delta/T_\Delta$ is pro-cyclic (maybe trivial). Hence we have the following cases.

Case $T_\Delta = \Delta$. Then $T_\Delta \cong (\mathbb{Z}/\ell)^\delta$, and by Lemma 2 (3), it follows that $\delta_v \ge \delta$. Since $\mathrm{td}(K|k) \ge r_v \ge \delta_v$, we finally get $\mathrm{td}(K|k) \ge \delta$.

Case $\Delta/T_\Delta$ is non-trivial. Then $\Delta/T_\Delta \cong \mathbb{Z}/\ell$ and $T_\Delta \cong (\mathbb{Z}/\ell)^{\delta-1}$. So the image of $\Delta$ in $\overline{\Pi}_{Kv}$ is non-trivial, hence $\overline{\Pi}_K$ is non-trivial. Since $kv$ is algebraically closed, we must have $Kv \neq kv$. Equivalently, $\mathrm{td}_v > 0$. Proceeding as above, we also have $r_v \geq (\delta - 1)$, hence finally: $\mathrm{td}(K|k) \geq r_v + \mathrm{td}_v > (\delta - 1)$ and $\mathrm{td}(K|k) \geq \delta$.

We now show the converse inequality. Using Pop [Pop06] Fact 5.6, one constructs valuations $v$ of $K$ such that $r_v = \mathrm{td}(K|k) =: d$. If $v$ is such a valuation, then $\dim(vK/\ell) = d$. Hence by Lemma 2 (3), $T_v \cong (\mathbb{Z}/\ell)^d$; and $T_v$ is a c.l. closed subgroup of $\overline{\Pi}_K$.

(2) By Proposition 15, it follows that $T_v, Z_v$ satisfy the properties asked for $T, Z$ in (2). For the converse assertion, let $T \subseteq Z$ be closed subgroups of $\overline{\Pi}_K$ satisfying the conditions (i) and (ii). Then $T \cong (\mathbb{Z}/\ell)^r$, and $Z$ contains closed subgroups $\Delta \cong (\mathbb{Z}/\ell)^d$ with $d > r$. And notice that the fact that $T''$ is the center of $Z''$ is equivalent to the fact that $T$ is the unique maximal subgroup of $Z$ such that $T$ and $(T, Z)$ are c.l.

*Step 1.* Consider a maximal c.l. subgroup $\Delta \cong (\mathbb{Z}/\ell)^d$ of $Z$. Since $T$ is by hypothesis a c.l. subgroup of $Z$ such that $(T, Z)$ is c.l. too, it follows that the closed subgroup $T_1$ of $Z$ generated by $T$ and $\Delta$ is a c.l. closed subgroup of $Z$. Hence by the maximality of $\Delta$ it follows that $T_1 \subseteq \Delta$, hence $T \subseteq \Delta$.

*Step 2.* Applying Proposition 12, let $v_0$ be the valuation of $K$ deduced from the data $(T, Z)$. Hence $Z \subseteq Z_{v_0}$, and $T = Z \cap T_{v_0}$. Let $\Lambda$ be the fixed field of $T$ in $K'$. Then we obviously have $K^T_{v_0} \subseteq \Lambda$. Let $v$ be the $\mathbb{Z}/\ell$-abelian $\Lambda$-core of $v_0$. We will eventually show that $v$ is an almost quasi $r$-divisor of $K$, and that $Z = Z_v$, $T = T_v$, thus concluding the proof. Note that $T \subseteq T_{v_0}$ implies $K^T_{v_0} \subseteq \Lambda$, and therefore $\Lambda v_0' = (Kv_0)'$. Hence by Proposition 6 (1), the same is true for the $\mathbb{Z}/\ell$-abelian $\Lambda$-core $v$ of $v_0$, and therefore we have $\Lambda v' = (Kv)'$.

*Step 3.* By Proposition 12 applied to $\Delta$, it follows that there exists a valuation $v_\Delta$ of $K$ such that $\Delta \subseteq Z_{v_\Delta}$. Moreover, by the discussion in the proof of assertion (1) above, it follows that $v_\Delta$ is defectless, and one of the following holds: Either $r_{v_\Delta} = d$, and moreover, in this case $\Delta = T_{v_\Delta}$, thus $\Delta$ consists of inertia elements only. Or $r_{v_\Delta} = d - 1$, and in this case we have $T_{v_\Delta} \cong (\mathbb{Z}/\ell)^{d-1}$, and since $\Delta$ is a c.l. subgroup of $\overline{\Pi}_K$, and $T \subseteq \Delta$ consists of inertia elements only, it follows that $T \subseteq T_{v_\Delta}$. Finally, $\Delta$ contains non inertia elements of $\overline{\Pi}_K$.

Let $w$ be the $\mathbb{Z}/\ell$-abelian $\Lambda$-core of $v_\Delta$. Since $T \subseteq T_{v_\Delta}$, we must have $K^T_{v_\Delta} \subseteq \Lambda$. Hence $\Lambda v'_{\ \Delta} = (Kv_\Delta)'$. But then by Proposition 6 (1), the same is true for $w$, i.e., $\Lambda w' = (Kw)'$.

*Claim.* $v = w$

We argue by contradiction. Suppose that $v \neq w$, and first suppose that even $v > w$. Since by the conclusion of Step 2 we have that $\Lambda v' = (Kv)'$ and $v$ equals its $\mathbb{Z}/\ell$-abelian $\Lambda$-core, it follows by Proposition 6 (1) that $\Lambda w' \neq (Kw)'$. But this contradicts the conclusion of Step 3.

Second, suppose that $v < w$. Then reasoning as above, we contradict the fact that $\Lambda v' = (Kv)'$. The claim is proved.

Now since $w$ is a coarsening of $v_\Delta$, and the latter valuation is defectless, the same is true for $w$, thus for $v = w$. We next claim that $v$ is an almost quasi $r$-divisor. Indeed, we have:

- First, $T \subseteq T_w = T_v$, and since $T \cong (\mathbb{Z}/\ell)^r$, it follows that $T_v \cong (\mathbb{Z}/\ell)^\delta$ for some $r \leq \delta \leq d$.

- Second, since $v \leq v_0$, it follows that $Z_{v_0} \subseteq Z_v$ Hence $Z \subseteq Z_v$, as $Z \subseteq Z_{v_0}$ by the definition of $v_0$.

- Moreover, $T_v$ is a c.l. subgroup of $Z_v$ such that $(T_v, Z_v)$ is c.l. too. In particular, $T$ is a c.l. subgroup of $Z_v$, and $(T, Z_v)$ is c.l. too.

The maximality of $Z$ and $T$ implies that $Z = Z_v$ and $T = T_v$, as claimed. □

## 10.5 A characterization of almost $r$-divisorial subgroups

**Definition 18.** We say that an almost quasi $r$-divisorial valuation of $K|k$ is an **almost $r$-divisorial valuation** or an **almost prime $r$-divisor** of $K|k$, if $v$ is trivial on $k$. We will further say that a closed subgroup $Z$ of $\overline{\Pi}_K$ endowed with a closed subgroup $T \subset Z$ is an **almost divisorial $r$-subgroup** of $\overline{\Pi}_K$, if there exists an almost prime $r$-divisor $v$ of $K|k$ such that $T = T_v$ and $Z = Z_v$.

We now show that using the information encoded in "sufficiently many" 1-dimensional projections, one can characterize the almost $r$-divisorial subgroups among all the almost quasi $r$-divisorial subgroups of $\overline{\Pi}_K$. See Pop [Pop06], especially Fact 4.5 for more details. Let us recall that for $t \in K$ a non-constant function, we denote by $\kappa_t$ the relative algebraic closure of $k(t)$ in $K$, and that the canonical (surjective) projection $pr_t : \overline{\Pi}_K \to \overline{\Pi}_{\kappa_t}$ is called a **one dimensional projection** of $\overline{\Pi}_K$.

**Theorem 19.** *Let $K|k$ be a function field as usual with* $\mathrm{td}(K|k) > r > 0$. *Then for a given almost quasi $r$-divisorial subgroup $Z \subseteq \overline{\Pi}_K$, the following assertions are equivalent:*

*(i) $Z$ is an almost $r$-divisorial subgroup of $\overline{\Pi}_K$.*
*(ii) There is an element $t \in K \backslash k$ such that $p_t(Z) \subseteq \overline{\Pi}_{\kappa_t}$ is an open subgroup.*

*Proof.* The proof is word-by-word identical with the one of Pop [Pop06], Proposition 5.6, and therefore we will omit the proof here. □

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