**Abstract**

We generalize the \( \mathbb{Z}/p \) metabelian birational \( p \)-adic section conjecture for curves, as introduced and proved in Pop [P2], to all complete smooth varieties, provided \( p > 2 \). [The condition \( p > 2 \) seems to be of technical nature only, and it might be removable.]

1. Introduction

The (birational) \( [p\text{-adic}] \) section conjecture (SC) originates from Grothendieck [G1], [G1], see rather [GGA], and weaker/conditional forms of the SC are a part of the local theory in anabelian geometry, see e.g. Faltings [Fa] and Szamuely [Sz]. In spite of serious efforts to tackle the SC, only the full Galois birational \( p \)-adic SC is completely resolved, see Koenigsmann [Ko2] for the case of curves, and Stix [St] for higher dimensional varieties. On the other hand, a much stronger form of the birational \( p \)-adic SC for curves, to be precise, the \( \mathbb{Z}/p \) metabelian birational \( p \)-adic SC for curves was proved in Pop [P2]. The aim of this note is to prove a similarly strong result for the higher dimensional varieties, at least in the case \( p > 2 \).

For reader’s sake and to make the presentation self contained (to some extent), we begin by recalling a few notations and well known facts, see e.g. the Introduction in [P2]. First, for an arbitrary (perfect) base field \( k \), and complete smooth geometrically integral \( k \)-varieties \( X \), let \( K = k(X) \) be the function field of \( X \). Let \( \tilde{X} \to X \) be the normalization of \( X \) in the field extension \( K \hookrightarrow \tilde{K} \). For \( x \in X \) and \( \tilde{x} \in \tilde{X} \) above \( x \), let \( T_x \subseteq Z_x \) be the inertia/decomposition, groups of \( \tilde{x}|x \), and \( G_x := \text{Aut}(\kappa(\tilde{x})|\kappa(x)) \) be the residual automorphism group. By decomposition theory, one has a canonical exact sequence of Galois groups:

\[
1 \to \text{Gal}(\tilde{K}|K\tilde{k}) \twoheadrightarrow \text{Gal}(\tilde{K}|K) \xrightarrow{\overline{p}_K} \text{Gal}(\tilde{k}|k) \to 1.
\]

Let \( \tilde{X} \to X \) be the normalization of \( X \) in the field extension \( K \hookrightarrow \tilde{K} \). For \( x \in X \) and \( \tilde{x} \in \tilde{X} \) above \( x \), let \( T_x \subseteq Z_x \) be the inertia/decomposition, groups of \( \tilde{x}|x \), and \( G_x := \text{Aut}(\kappa(\tilde{x})|\kappa(x)) \) be the residual automorphism group. By decomposition theory, one has a canonical exact sequence

\[(*)\]

\[
1 \to T_x \to Z_x \to G_x \to 1.
\]

Next suppose that \( x \) is \( k \)-rational, i.e., \( \kappa(x) = k \). Since \( \tilde{k} \subseteq \kappa(\tilde{x}) \), the projection \( Z_x \xrightarrow{\overline{p}_K} \text{Gal}(\tilde{k}|k) \) gives rise to a canonical surjective homomorphism \( G_x \to \text{Gal}(\tilde{k}|k) \), which in general is not injective. And if \( \kappa(\tilde{x})|\tilde{k} \) is purely inseparable, then the canonical homomorphism \( G_x \to \text{Gal}(\tilde{k}|k) \) is an isomorphism. Hence if the exact sequence \((*)\) splits, then \( \overline{p}_K \) has sections \( \tilde{s}_x : \text{Gal}(\tilde{k}|k) \to Z_x \subseteq \text{Gal}(\tilde{K}|K) \), which we call sections above \( x \); and notice that the conjugacy classes of sections \( \tilde{s}_x \) above \( x \) build a “bouquet”, which is in a canonical bijection with the (non-commutative) continuous cohomology

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pointed set $H^1_{\text{cont}}(\text{Gal}(\bar{k}|k), T_x)$ defined via the split exact sequence (*).

Parallel to the case of points $x \in X$, one has a similar situation for $k$-valuations $v$ of $K$ as follows. For any prolongation $\tilde{v}$ of $v$ to $\bar{K}$, we denote by $T_v \subseteq Z_v$ the inertia/decomposition groups of $\tilde{v}|v$, and by $G_v = Z_v/T_v$ the residual automorphism group. As above, if $\kappa(v) = k$ and $\kappa(\tilde{v})|k$ is purely inseparable, one has: First, the canonical homomorphism $G_v \to \text{Gal}(\bar{k}|k)$ is an isomorphism. Second, if the exact sequence $1 \to T_v \to Z_v \to G_v \to 1$ splits, then $\bar{p}_v : \text{Gal}(\bar{K}|K) \to \text{Gal}(\bar{k}|k)$ has a section $\tilde{s}_v : \text{Gal}(\bar{k}|k) \to Z_v \subseteq \text{Gal}(\bar{K}|K)$, which we call a section above $v$. And the conjugacy classes of sections $\tilde{s}_v$ above $v$ build a “bouquet” which is in a canonical bijection with the (non-commutative) continuous cohomology pointed set $H^1_{\text{cont}}(\text{Gal}(\bar{k}|k), T_v)$ defined via the canonical split exact sequence above.

Finally, if $\bar{K}|K$ contains a separable closure $K^s|K$ of $K$, hence $\bar{k} \supseteq k^s$ too, then $\kappa(s)^s \subseteq \kappa(\tilde{x}), \kappa(\tilde{v})$ and $G_x$ and $G_v$ are the absolute Galois groups of $\kappa(x)$, respectively $\kappa(v)$. Further, in this situation, the canonical exact sequence $1 \to T_v \to Z_v \to G_v \to 1$ is split, see e.g., [KPR]. Thus if $\kappa(v) = k$, sections above $v$ exist. In particular, if $x \in X(k)$ is a $k$-rational point, then choosing $v$ such that $\kappa(x) = \kappa(v)$, it follows that sections above $x$ exist as well, because every section above $v$ is a section above $x$ as well. We mention though that in general the bouquet of sections above $x$ is much richer than the one of sections above $v$. Indeed, by general decomposition theory one has $T_v \subseteq T_x$, and the canonical map $H^1_{\text{cont}}(\text{Gal}(\bar{k}|k), T_v) \to H^1_{\text{cont}}(\text{Gal}(\bar{k}|k), T_x)$ is a strict inclusion in general.

• Next let $p$ be a fixed prime number. We denote by $K'|K$ the (maximal) $\mathbb{Z}/p$ elementary abelian extension of $K$, and by $K''|K'$ the maximal $\mathbb{Z}/p$ elementary abelian extension of $K'$ (in some fixed algebraic closure of $K$). Then $K''|K$ is a Galois extension, which we call the $\mathbb{Z}/p$ metabelian extension of $K$, and its Galois group $\text{Gal}(K''|K)$ is called the metabelian Galois group of $K$. Note that $k'|\bar{K} \cap K'$, and $k''|\bar{K} \cap K''$ are the $\mathbb{Z}/p$ elementary abelian extension, respectively the $\mathbb{Z}/p$ metabelian extension, of $k$. Finally, consider the canonical surjective projections:

$$pr'_K : \text{Gal}(K'|K) \to \text{Gal}(k'|k), \quad pr''_K : \text{Gal}(K''|K) \to \text{Gal}(k''|k).$$

We will say that a group theoretical (continuous) section $s' : \text{Gal}(k'|k) \to \text{Gal}(K'|K)$ of $pr'_K$ is liftable, if there exists a section $s'' : \text{Gal}(k''|k) \to \text{Gal}(K''|K)$ of $pr''_K$ which lifts $s'$ to $\text{Gal}(k''|k)$.

Note that if $p \neq \text{char}$, and the $p$th roots of unity $\mu_p$ are contained in $k$, hence in $K$, then by Kummer Theory we have: $K' = K[\sqrt[p]{K}]$, and $K'' = K'[\sqrt[p]{K'}]$, and similarly for $k$.

**Theorem A.** In the above notations, let $k|\mathbb{Q}_p$ be finite with $\mu_p \subset k$. Then the following hold:

1) Every $k$-rational point $x \in X$ gives rise to a bouquet of conjugacy classes of liftable sections $s'_x : \text{Gal}(k'|k) \to \text{Gal}(K'|K)$ above $x$, which is in bijection with $H^1(\text{Gal}(k'|k), T_x)$.

2) Let $p > 2$, and $s' : \text{Gal}(k'|k) \to \text{Gal}(K'|K)$ be a liftable section. Then there exists a unique $k$-rational point $x \in X$ such that $s'$ equals one of the sections $s'_x$ as defined above.

Actually one can reformulate the question addressed by Theorem A in terms of $p$-adic valuations, and get the following stronger result, see Section 2, C), for notations, definitions, and a few facts on (formally) $p$-adic valuations $v$, e.g., the $p$-adic rank $d_v$ of $v$, and $p$-adically closed fields, respectively Ax–Kochen [A–K] and Prestel–Roquette [P–R], for proofs:

**Theorem B.** Let $k$ be a $p$-adically closed field with $p$-adic valuation $v$ of $p$-adic rank $d_v$, and suppose that $\mu_p \subset k$. Let $K|k$ be an arbitrary field extension. Then the following hold:

1) Let $w$ be a $p$-adic valuation of $K$ of $p$-adic rank $d_w = d_v$. Then $w$ prolongs $v$ to $K$, and gives rise to a bouquet of conjugacy classes of liftable sections $s'_w : \text{Gal}(k'|k) \to \text{Gal}(K'|K)$ above $w$. 

Birational Section Conjecture

2) Let \( p > 2 \), and \( s' : \text{Gal}(k'|k) \to \text{Gal}(K'|K) \) be a liftable section. Then there exists a unique \( p \)-adic valuation \( w \) of \( K \) of \( p \)-adic rank \( d_w = d_v \) such that \( s' = s'_w \) for some \( s'_w \) as above.

Remark/Definition. As mentioned in Pop [P2], the condition \( \mu_p \subset k \) is a necessary condition in the above theorems. Nevertheless, as mentioned in loc.cit., if \( \mu_p \) is not contained in the base field, assertions similar to Theorems A and B above hold in the following form: Let \( l|\mathbb{Q}_p \) be some finite extension, \( Y \rightarrow l \) a complete geometrically integral smooth variety with function field \( L = k(Y) \). Let \( k|l \) be a finite Galois extension with \( \mu_p \subset k \). Setting \( K := Lk \), consider the field extensions \( K'|K \leftrightarrow K''|k \) and \( k'|k \leftrightarrow k''|k \) as above. Then \( k' = K' \cap l \) and \( k'' = K'' \cap l \), and \( K'|L \) and \( K''|L \), as well as \( k'|l \) and \( k''|l \) are Galois extensions too, and one gets surjective canonical projections

\[
pr'_L : \text{Gal}(K'|L) \to \text{Gal}(k'|l), \quad pr''_L : \text{Gal}(K''|L) \to \text{Gal}(k''|l).
\]

In these notations and context we will say that a section \( s'_L : \text{Gal}(k'|l) \to \text{Gal}(K'|L) \) of \( pr'_L \) is liftable, if there exists a section \( s''_L : \text{Gal}(k''|l) \to \text{Gal}(K''|L) \) of \( pr''_L \) which lifts \( s'_L \).

This being said, one has the following extensions of Theorem A and Theorem B:

Theorem A'. In the above notations and hypothesis, the following hold:

1) Every \( l \)-rational point \( y \in Y \) gives rise to a bouquet of conjugacy classes of liftable sections \( s'_y : \text{Gal}(k'|l) \to \text{Gal}(K'|L) \) above \( y \), which is in bijection with \( H^1(\text{Gal}(k'|l), T_y) \).

2) Let \( p > 2 \) and \( s'_L : \text{Gal}(k'|l) \to \text{Gal}(K'|L) \) be a liftable section. Then there exists a unique \( l \)-rational point \( y \in Y \) such that \( s'_L \) equals one of the sections \( s'_y \) as defined above.

Theorem B'. Let \( l \) be a \( p \)-adically closed field with \( p \)-adic valuation \( v \), and let \( L|l \) be an arbitrary field extension. Then in the above notations the following hold:

1) Let \( w \) be a \( p \)-adic valuation of \( L \) with \( d_w = d_v \). Then \( w \) prolongs \( v \) to \( L \), and gives rise to a bouquet of conjugacy classes of liftable sections \( s'_w : \text{Gal}(k'|l) \to \text{Gal}(K'|L) \) above \( w \).

2) Let \( p > 2 \) and \( s'_L : \text{Gal}(k'|l) \to \text{Gal}(K'|L) \) be a liftable section. Then there exists a unique \( p \)-adic valuation \( w \) of \( L \) such that \( d_w = d_v \), and \( s'_L \) equals one of the sections \( s'_w \) as above.

Remark. As mentioned in Pop [P2], the \( \mathbb{Z}/p \) metabelian form of the birational \( p \)-adic SC for curves implies the corresponding full Galois SC, which was proved in Koenigsmann [Ko2]. The same hold correspondingly for higher dimensional varieties, provided \( p > 2 \), thus implying Stix [St] result in this case. Since the proof of the implication under discussion in the case of general varieties is word-by-word the same as that from [P2] loc.cit., we will not reproduce it here.

An interesting application of the results and techniques developed here is the following fact concerning the \( p \)-adic Section Conjecture for varieties: Let \( k|\mathbb{Q}_p \) be a finite extension, and \( X \) a complete smooth \( k \)-variety. Then there exists a finite effectively computable family of finite geometrically \( \mathbb{Z}/p \) elementary abelian (ramified) covers \( \varphi_i : X_i \to X \), \( i \in I \), satisfying:

i) \( \bigcup_i \varphi_i(X_i(k)) = X(k) \), i.e., every \( x \in X(k) \) “survives” in at least one of the covers \( X_i \to X \).

ii) A section \( s : G_k \to \pi_1(X, \ast) \) can be lifted to a section \( s_i : G_k \to \pi_1(X_i, \ast) \) for some \( i \in I \) if and only if \( s \) arises from a \( k \)-rational point \( x \in X(k) \) in the way described above.

The main technical tools for the proof of the above theorems are:

- The techniques developed in Pop [P2] (which refine facts/methods initiated in [P1]).
- The theory of rigid elements, as developed by several people: Ware [Wa], Arason–Jacob–Ware [AJW], Koenigsmann [Ko1], Efrat [Ef], etc. But see rather TOPAZ [To].
As a final remark, we notice that the condition \( p > 2 \) in the results above originates from the weaker results about recovering valuations from rigid elements in the case \( p = 2 \). This technical condition might be removable, but some new ideas/techniques might be necessary to do so, see the comment at the beginning of the proof of assertion 2) of Theorem B in section 3.

2. Reviewing a few known facts

For reader’s sake, in this section we review a few known facts about valuation theory, decomposition theory, and (formally) \( p \)-adic fields, but do not reproduce proofs.

A) **Generalities about valuations and their Hilbert decomposition theory**

For an arbitrary field \( K \) and an arbitrary valuation \( v \) of \( K \), we denote usually by \( O_v, m_v \) the valuation ring/ideal of \( v \), by \( vK = K^*/O^* \) the value group of \( v \), and by \( Kv = O_v/m_v := \kappa(v) \) the residue field of \( v \). Further, \( U_v^1 := 1 + m_v \subseteq U_v \) denote the groups of principal \( v \)-units, respectively \( v \)-units. One has the following canonical exact sequences:

\[
1 \to m_v \to O_v \to Kv \to 0 \quad \text{and} \quad 1 \to U_v^1 \to O^*_v \to (Kv)^* \to 1.
\]

The set of ideals of \( O_v \) is totally ordered with respect to inclusion. The subrings \( O_1 \subseteq K \) with \( O_v \subseteq O_1 \) are precisely the localizations \( O_1 := (O_v)_{m_1} \) with \( m_1 \in \text{Spec}(O_v) \), and moreover, \( m_1 \subset O_v \), and \((O_v)_{m_1} \) is a valuation ring with valuation ideal \( m_1 \). Further, if \( v_1 \) is the corresponding valuation of \( K \), then \( O_0 := O_v/m_1 \) is a valuation ring of \( Kv_1 \) with valuation ideal \( m_0 := m_v/m_1 \), say of a valuation \( v_0 \) of \( Kv_1 \). We say that \( v_1 \) is a **coarsening** of \( v \), and denote \( v_1 \preceq v \) and \( v_0 := v/v_1 \).

Conversely, if \( v_i \) is a valuation of \( K \) and \( v_0 \) is a valuation on the residue field \( Kv_1 \), then the preimage of the valuation ring \( O_{v_0} \subseteq Kv_1 \) under \( O_{v_1} \to Kv_1 \) is a valuation ring \( O \subseteq O_{v_1} \) having as valuation ideal the preimage \( m \subseteq O \) of \( m_{v_0} \). Hence if \( v \) is the valuation defined by \( O \) on \( K \), then \( Kv = (Kv_1)v_0 \) and one has a canonical exact sequence of totally ordered groups:

\[
0 \to v_0(Kv_1) \to vK \to v_1K \to 0.
\]

The relation between coarsening and decomposition theory is as follows. Let \( \tilde{K}|K \) be a Galois extension, and \( \tilde{v}|v \) be a prolongation of \( v \) to \( \tilde{K} \). Then the coarsenings \( \tilde{v}_i \) of \( \tilde{v} \) are in a canonical bijection with the coarsenings \( v_i \) of \( v \) via \( O_{\tilde{v}_i} := O_{\tilde{v}_i} \cap K \), thus \( O_{\tilde{v}_i} = O_{\tilde{v}} : O_{v_i} \). Let \( \tilde{v}_i \) be given coarsenings of \( \tilde{v}|v \) and \( K\tilde{v}_i |Kv_1 \) is the corresponding residue field extension. Then \( \tilde{v}_0 := \tilde{v}/\tilde{v}_1 \) is canonically a prolongation of \( v_0 := v/v_1 \).

**FACT 1.** Let \( T_\tilde{v} \subseteq \tilde{Z}_\tilde{v} \) and \( T_{\tilde{v}_1} \subseteq \tilde{Z}_{\tilde{v}_1} \) be the corresponding inertia/decomposition groups, and set \( G_{\tilde{v}_1} = \text{Aut}(K\tilde{v}_1 |Kv_1) \). Then one has a canonical exact sequence \( 1 \to T_{\tilde{v}_1} \to \tilde{Z}_{\tilde{v}_1} \to G_{\tilde{v}_1} \to 1 \), and the inertia/decomposition groups satisfy:

a) \( Z_{\tilde{v}} \subseteq Z_{\tilde{v}_1} \) and \( T_{\tilde{v}} \supseteq T_{\tilde{v}_1} \). Further, \( T_{\tilde{v}_1} \) is a normal subgroup of \( Z_{\tilde{v}} \).

b) \( \text{Via } 1 \to T_{\tilde{v}_1} \to Z_{\tilde{v}_1} \to G_{\tilde{v}_1} = Z_{\tilde{v}_1}/T_{\tilde{v}_1} \to 1 \), one has that \( T_{\tilde{v}_0} = T_{\tilde{v}}/T_{\tilde{v}_1} \) and \( Z_{\tilde{v}_0} = Z_{\tilde{v}}/T_{\tilde{v}_1} \).

B) **Hilbert decomposition in elementary abelian extensions**

Let \( K \) be a field of characteristic prime to \( p \) containing \( \mu_n \), where \( n = p^e \) is a power of the prime number \( p \), and let \( \tilde{K} = K[\sqrt[n]{K}] \) be the maximal \( \mathbb{Z}/n \) elementary abelian extension of \( K \). Let \( v \) be a valuation of \( K \), and \( \tilde{v} \) some prolongation of \( v \) to \( \tilde{K} \), and \( V_\tilde{v} \subseteq T_{\tilde{v}} \subseteq \tilde{Z}_{\tilde{v}} \) be the ramification, the inertia, and the decomposition, groups of \( \tilde{v}|v \), respectively. We remark that because Gal(\( \tilde{K}|K \)) is commutative, the groups \( V_\tilde{v}, T_{\tilde{v}}, \) and \( Z_{\tilde{v}} \) depend on \( v \) only. Therefore we will simply denote them by \( V_v, T_v, \) and \( Z_v \). Finally, we denote by \( K^Z \subseteq K^T \subseteq K^V \) the corresponding fixed fields in \( \tilde{K} \). One has the following, see e.g. Pop [P2], Section 2 (where the case \( n = p \) is dealt with; but the proof is absolutely similar for general \( n = p^e \) and we will not reproduce the details here again):
FACT 2. In the above notations, the following hold:

1) Let \( U \) be a field of characteristic \( p \). Then \( \sqrt[p]{U} \) is a unique valuation ring of \( U \), and \( U^\sqrt[p]{U} \) is the unique maximal proper coarsening of \( U \). Note that setting \( \sqrt[p]{U} \) be the Henselization of \( U \) with respect to \( v \).

2) Let \( v \) be a valuation of \( K \) with valuation ring \( \mathcal{O}_v \). Then \( \mathcal{O}_v := \mathcal{O}_v[1/p] \) is the valuation ring of the unique maximal proper coarsening \( v \), of which is called the canonical coarsening of \( v \). Note that setting \( k_v := kv \), and \( v_0 := v/v_1 \) the corresponding valuation on \( k_v \) have: \( v_0 \) is a valuation of \( k_v \) with \( e_{v_0} = e_v \) and \( f_{v_0} = f_v \), hence \( d_{v_0} = d_v \), and moreover, \( v_0 \) is a discrete valuation of \( k_v \). In particular, the following hold:
   a) \( v \) has rank one iff \( v_1 \), is the trivial valuation iff \( v = v_0 \).
   b) Giving a \( p \)-adic valuation \( v \) of a field \( k \) is equivalent to giving a place \( p \) of \( k \) with values in a finite extension \( k_v \) of \( Q_p \) such that the residue field \( k_v := kp \) is dense in \( k_v \), and \( k_v \) has ramification index \( e_v \) and residual degree \( f_v \).
   c) If \( v_1 < v \) is the strict coarsening of \( v \), then \( v_1 \leq v \), and the quotient valuation \( v/v_1 \) on the residue field \( k_v \) is a \( p \)-adic valuation with \( e_{v/v_1} = e_v \) and \( f_{v/v_1} = f_v \), thus \( d_{v/v_1} = d_v \).
      (Actually, \( \kappa(v_1/v) \cong kv \), and \( \kappa(v_1/v) \cong kv \) canonically.)

3) Let \( k \) be a field of characteristic \( p \). Then \( k^{\sqrt[p]{k}} \) is a unique valuation ring of \( k \) with valuation ring \( \mathcal{O}_v \). Then \( \mathcal{O}_v := \mathcal{O}_v[1/p] \) is the valuation ring of the unique maximal proper coarsening \( v \), of which is called the canonical coarsening of \( v \). Note that setting \( k_v := kv \), and \( v_0 := v/v_1 \) the corresponding valuation on \( k_v \) have:
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   c) If \( v_1 < v \) is the strict coarsening of \( v \), then \( v_1 \leq v \), and the quotient valuation \( v/v_1 \) on the residue field \( k_v \) is a \( p \)-adic valuation with \( e_{v/v_1} = e_v \) and \( f_{v/v_1} = f_v \), thus \( d_{v/v_1} = d_v \).
      (Actually, \( \kappa(v_1/v) \cong kv \), and \( \kappa(v_1/v) \cong kv \) canonically.)

4) A field \( k \) is called (formally) \( p \)-adically closed, if \( k \) carries a \( p \)-adic valuation \( v \) such that for every finite extension \( k' \) of \( k \) one has: If \( v \) has a prolongation \( v' \) to \( k \) with \( d_v \), then \( k' = k \). One has the following characterization of the \( p \)-adically closed fields: For a field \( k \) endowed with a \( p \)-adic valuation \( v \), and its canonical coarsening \( v_1 \), the following are equivalent:
i) $k$ is $p$-adically closed with respect to $v$.

ii) $v$ is Henselian, and $v,k$ is divisible (maybe trivial).

iii) $v_1$ is Henselian, $v_1,k$ is divisible (maybe trivial), and the residue field $k_0 := kv_1$ is relatively algebraically closed in its $v_0 = v/v_1$ completion $k_0$ (which itself is a finite extension of $\mathbb{Q}_p$).

Further, the $p$-adic valuation of a $p$-adically closed field is definable and unique.

5) Finally, for every field $k$ endowed with a $p$-adic valuation $v$, there exist $p$-adic closures $\hat{k}, \hat{v}$ such that $d_{\hat{v}} = d_v$. Moreover, the space of the $k$-isomorphy classes of $p$-adic closures of $k, v$ has a concrete description as follows: Let $v_1$ be the canonical coarsening of $v$, and $k_0|\mathbb{Q}_p$ the completion of the residue field of $k_0 = kv_1$ with respect to the discrete valuation $v_0 = v/v_1$.

Recalling the canonical exact sequence $1 \to I_{v_1} \to D_v \xrightarrow{pr} G_{k_0} \to 1$, one has that the space of the isomorphy classes of $p$-adic closures of $k, v$ is in bijection with the space of conjugacy classes of sections of $pr$, thus with $H^1_{cont}(G_{k_0}, I_{v_1})$.

6) In the above notations, the following hold:

a) Let $k, v$ be a $p$-adically closed field. Then $k_0 = kv_1$ is $p$-adically closed (with respect to $v_0$), and $k^{abs}$ is actually the relative algebraic closure of $\mathbb{Q}$ in $k_0$. Further, $\hat{k} = k\hat{\mathbb{Q}}$.

b) The elementary equivalence class of a $p$-adically closed field $k$ is determined by both: The absolute subfield $k^{abs} := k \cap \mathbb{Q} = k_0 \cap \mathbb{Q}$ of $k$, and the completion $k_0$ of $k_0 = kv_1$ with respect to $v_0$ (which equals the completion of $k^{abs}$ with respect to $v_0$ as well).

c) If $N$ is $p$-adically closed with respect to the $p$-adic valuation $w$, and $k \subseteq N$ is a subfield which is relatively closed in $N$, then $k$ is $p$-adically closed with respect to $v := w|_k$, and $v$ and $w$ have equal $p$-adic ranks, and $N$ and $k$ are elementary equivalent.

d) If $N|k$ is an extension of $p$-adically closed fields of the same rank, the following hold:

- $\hat{k}|k \to N\hat{k}$ defines a bijection between the algebraic extensions $\hat{k}|k$ of $k$ and those of $N$.

- The canonical projection $G_N \to G_k$ is an isomorphism.

e) In particular, if $L|l$ is an extension of $p$-adically closed fields of the same rank, in the notations from the Introduction, the following canonical projections are isomorphisms:

$$\tag{\dagger} \text{pr}_L' : \text{Gal}(K'|L) \to \text{Gal}(k'|l), \quad \text{pr}_L'' : \text{Gal}(K''|L) \to \text{Gal}(k''|l).$$

D) Valuations and rigid elements

We recall the result Arason–Elman–Jacob [AEJ], Theorem 2.16, see also Koenigsmann [Ko1], Ware [Wa], Efrat [Ef], and rather Topaz [To] for much more about this. The point here is that one can recover valuations of an arbitrary field $K$ from particular subgroups $T \subset K^\times$ as follows: Let $T \subset K^\times$ be a subgroup with $-1 \in T$. We say that $x \in K^\times \setminus T$ is $T$-rigid, if $1 + x \in T \cup xT$; and by abuse of language, we say that that $K$ is $T$-rigid, if all $x \in K^\times \setminus T$ are $T$-rigid.

**Theorem 3** (Arason–Elman–Jacob). In the above notations, let $T \subset K^\times$ be a subgroup with $-1 \in T$ such that $K$ is $T$-rigid. Then there exist a valuation $v$ of $K$ whose valuation ideal $m_v$ satisfies $1 + m_v \subseteq T$, and whose valuation ring $\mathcal{O}_v$ has the property that $|\mathcal{O}_v^*/(T \cap \mathcal{O}_v^*)| \leq 2$.

3. Proof of Theorem B

To 1): Let $\hat{K}, \hat{w}$ be a $p$-adic closure of $K, w$. Then $\hat{w}$ prolongs $w$ and has $p$-adic rank $d_{\hat{w}} = d_w$, thus equal to $d_w$ by the fact that $d_w = d_v$. Therefore, since $k$ is $p$-adically closed, $k$ must be relatively algebraically closed in $\hat{K}$. Conclude by applying relation $(\dagger)$ from Section 2, C), 6), e), with $l := k$ and $L := \hat{K}$, and taking into account that the isomorphisms $\text{Gal}(K''|\hat{K}) \to \text{Gal}(k''|k)$ factors through $\text{Gal}(K''|K) \to \text{Gal}(k''|k)$, thus gives rise to a liftable section of $\text{Gal}(K''|K) \to \text{Gal}(k''|k)$. 

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By Kummer theory, \( pr'_K : \text{Gal}(K'|K) \rightarrow \text{Gal}(k'|k) \) is Pontrjagin dual to the canonical embedding \( k^\times/p \rightarrow K^\times/p \). Second, given a liftable section \( s' : \text{Gal}(k'|k) \rightarrow \text{Gal}(K'|K) \) of \( pr'_K \), it follows by Kummer theory that the Pontrjagin dual of \( s' : \text{Gal}(k'|k) \rightarrow \text{Gal}(K'|K) \) is a surjective projection \( K^\times/p \rightarrow k^\times/p \), whose kernel \( \Sigma/p \subset K^\times/p \) is a complement of \( k^\times/p \subset K^\times/p \). That means, \( s' \) gives rise canonically to a presentation of \( K^\times/p \) as a direct sum

\[
K^\times/p = \Sigma/p \cdot k^\times/p.
\]

For every \( k \)-subfield \( K_\alpha \subset K \) which is relatively algebraically closed in \( K \), one has a commutative diagram of surjective projections

\[
\begin{array}{ccc}
\text{Gal}(K'|K) & \longrightarrow & \text{Gal}(K''|K_\alpha) \\
\downarrow & & \downarrow \\
\text{Gal}(K'|K) & \longrightarrow & \text{Gal}(K'_\alpha|K_\alpha)
\end{array}
\]

and \( s' \) gives rise canonically to a liftable section \( s'_\alpha \) of \( pr'_\alpha : \text{Gal}(K'_\alpha|K_\alpha) \rightarrow \text{Gal}(k'|k) \), etc. In particular, one has corresponding canonical presentations as direct sums

\[
(\dagger) \quad K'_\alpha/p = \Sigma_\alpha/p \cdot k^\times/p
\]

defined by the liftable sections \( s'_\alpha : \text{Gal}(k'|k) \rightarrow \text{Gal}(K'_\alpha|K_\alpha) \) induced by \( s' : \text{Gal}(k'|k) \rightarrow \text{Gal}(K'|K) \).

CLAIM. In the above notions one has: \( \Sigma_\alpha/p = \Sigma/p \cap K^\times_\alpha/p \), thus \( \Sigma/p \) determines \( \Sigma_\alpha/p \).

Indeed, let \( D_\alpha := \text{im}(s'_\alpha) \subset \text{Gal}(K'_\alpha|K_\alpha) \). Then by the definition of \( s'_\alpha \), it follows that \( D_\alpha \) is the image of \( D := \text{im}(s') \) under the canonical projection \( \text{Gal}(K'|K) \rightarrow \text{Gal}(K'_\alpha|K_\alpha) \). In other words, by Pontrjagin duality, the projection \( K^\times_\alpha/p \rightarrow k^\times/p \) factors through the inclusion \( K^\times_\alpha/p \rightarrow K^\times/p \). Hence \( \Sigma_\alpha/p \) is mapped into \( \Sigma/p \) under \( K^\times_\alpha/p \hookrightarrow K^\times/p \), which proves the claim.

Now let \( T/p := \Sigma/p \cdot \mathcal{O}^\times_v/p \) and \( T \subset K^\times \) be the corresponding subgroup (thus containing the \( p^\text{th} \) powers in \( K^\times \)). Then for every \( k \)-subfield \( K_\alpha \subset K \) which is relatively algebraically closed in \( K \), by the remarks above one has that \( T_\alpha := T \cap K^\times_\alpha \subset K^\times_\alpha \) satisfies \( T_\alpha/p = \Sigma_\alpha/p \cdot \mathcal{O}^\times_v/p \).

Finally, let \( (K_\alpha)_\alpha \) be the family of all the \( k \)-subfields \( K_\alpha \subset K \) which are relatively algebraically closed in \( K \) and satisfy \( \text{tr.deg}(K_\alpha/k) = 1 \). Then by Theorem B of Pop [P2], for every subfield \( K_\alpha \), there exists a unique \( p \)-adic valuation \( w_\alpha \) of \( K_\alpha \) prolonging the \( p \)-adic valuation \( v \) of \( k \) to \( K_\alpha \) and having the same \( p \)-adic rank as \( v \). Our final aim is to show that there exists a (unique) \( p \)-adic valuation \( w \) of \( K \) such that \( w_\alpha \) is the restriction of \( w \) to \( K_\alpha \) for each \( K_\alpha \).

LEMMA 4. In the above notations, \( K_\alpha \) is \( T_\alpha \) rigid. Further, \( T = \cup_\alpha T_\alpha \), and \( K \) is \( T \)-rigid.

Proof. We first show that \( \mathcal{O}_{w_\alpha} \subset T_\alpha \). Indeed, let \( v \) be the \( p \)-adic valuation of \( k \), and further consider: First, the canonical coarsening \( v_1 \) of \( v \), and the canonical \( p \)-adic valuation \( v_0 := v/v_1 \) on the residue field \( k_0 := k v_1 \) of \( v_1 \). Second, for the \( p \)-adic valuation \( w_\alpha \) of \( K_\alpha \), let \( w_{\alpha 1} \) and \( w_{\alpha 0} := w_\alpha/w_{\alpha 1} \), and \( K_{\alpha 0} \) be correspondingly defined. Notice that \( w_{\alpha 1} \mid k = v \) implies that \( w_{\alpha 1} \mid 1 = v_1 \) and \( w_{\alpha 0} \mid k_0 = v_0 \). The following hold:

a) First, by Fact 2, it follows that \( \sqrt[\alpha]{1 + p^2 \mathbf{m}_{w_\alpha}} \) is contained in the decomposition field of \( w_\alpha \) over \( K \), which is actually the fixed field of \( Z_{w_\alpha} \) in \( K_\alpha \). Second, the fixed field of \( \text{im}(s') \) in \( K'_\alpha \) is, by the mere definitions, generated as a field extension of \( K \) by \( \sqrt[\alpha]{\Sigma_\alpha} \). Thus since \( \text{im}(s') \subset Z_{w_\alpha} \), it follows by Kummer theory that \( 1 + p^2 \mathbf{m}_{w_\alpha} \subset \Sigma_\alpha \).

b) Since, by the mere definition, one has \( \mathbf{m}_{w_{\alpha 1}} \subset \mathbf{m}_{w_\alpha} \) and \( p \) is invertible in \( \mathcal{O}_{w_{\alpha 1}} \), it follows that \( 1 + \mathbf{m}_{w_{\alpha 1}} \subset 1 + p^2 \mathbf{m}_{w_\alpha} \). Thus finally, \( 1 + \mathbf{m}_{w_{\alpha 1}} \subset \Sigma_\alpha \) as well.
c) Since $w_\alpha$ and $v$ have the same $p$-adic rank, it follows by the discussion in section 2, C, 5), that $w_\alpha$ and $v_0$ are discrete $p$-adic valuations of the same $p$-adic rank, hence $k_0$ is dense in $K_{\alpha_0}$.

Therefore, since $w_\alpha|v_0$ are discrete valuations, and $k_0$ is dense in $K_{\alpha_0}$ under $k_0 \hookrightarrow K_{\alpha_0}$, one has that $O_{w_\alpha}^x = O_{v_0}^x \cdot (1 + p^2m_{w_\alpha})$ and $K_{\alpha_0}^x = k_0^x \cdot (1 + p^2m_{w_\alpha})$ as well.

d) Since $K_{\alpha_0}^x = O_{w_\alpha}^x/(1 + m_{w_\alpha})$, $k_0^x = O_{v_1}^x/(1 + m_{v_1})$, and $1 + p^2m_{w_\alpha} = (1 + p^2m_{w_\alpha})/(1 + m_{w_\alpha})$, from the equality $K_{\alpha_0}^x = k_0^x \cdot (1 + p^2m_{w_\alpha})$ above, it follows that $O_{w_\alpha}^x = O_{v_1}^x \cdot (1 + p^2m_{w_\alpha})$.

e) Similarly, $O_{w_\alpha}^x = O_{w_\alpha}^x/(1 + m_{w_\alpha})$ and $O_{w_\alpha}^x = O_{v_1}^x/(1 + p^2m_{w_\alpha})$, imply $O_{w_\alpha}^x = O_{v_1}^x \cdot (1 + p^2m_{w_\alpha})$.

Hence since $O_{v_1}^x, 1 + p^2m_{w_\alpha} \subset T_\alpha$, one finally has $O_{w_\alpha}^x = O_{v_1}^x \cdot (1 + p^2m_{w_\alpha}) \subset T_\alpha$, as claimed.

We next show that $K_\alpha$ is $T_\alpha$-rigid. To do so, we first notice that by the discussion above, for any fixed element $\pi \in O_v$ of minimal positive value $1_v \in vK$, the following holds: Let $x \in O_{w_\alpha}^x$ be an arbitrary $w_{\alpha_1}$-unit. Then there exist $m \in \mathbb{Z}$, $\epsilon \in O_v^x$, $x_1 \in 1 + p^2m_{w_\alpha}$ such that

$$x = \pi^m \epsilon x_1.$$ (2)

Now let $x \in K_\alpha^x \setminus T_\alpha$ be given. Then one has the following possibilities:

1) $w_{\alpha_1}(x) > 0$. Then $1 + x$ is a principal $w_{\alpha_1}$-unit, and therefore, $1 + x \in \Sigma_\alpha$ by assertion b) above. Since $\Sigma_\alpha \subset T_\alpha$, we conclude that $1 + x \in T_\alpha$.

2) $w_{\alpha_1}(x) < 0$. Then $1 + x = x(1 + x^{-1})$. Since $w_{\alpha_1}(x^{-1}) > 0$, by the discussion above, it follows that $1 + x^{-1} \in T_\alpha$. Therefore, one finally has that $1 + x \in xT_\alpha$.

3) $w_{\alpha_1}(x) = 0$, or equivalently, $x \in O_{w_\alpha}^x$. Let $x = \pi^m \epsilon x_1$ be as given at (2) above. One has:

a) If $m > 0$, then $x \in \pi^m \cdot O_{w_\alpha}^x$, thus $1 + x \in O_{w_\alpha}^x$ as well. Hence by the relation (2) above, $1 + x = \eta_1 \cdot \eta_0$ for some $\eta_1 \in 1 + p^2m_{w_\alpha} \subset \Sigma_\alpha$, $\eta_0 \in O_v^x$. Thus finally, $1 + x \in T_\alpha$.

b) If $m < 0$, then $1 + x = x(1 + x^{-1})$, and $x^{-1}$ has value $-m > 0$. But then, by the first case above, $1 + x^{-1} \in T_\alpha$. Hence $1 + x = x(1 + x^{-1}) \in xT_\alpha$, thus $1 + x \in xT_\alpha$.

c) If $m = 0$, then $x \in O_{w_\alpha}^x \subset T_\alpha$, thus $x \notin K_\alpha^x \setminus T_\alpha$.

For the $T$-rigidity of $K$, let $x \in K \setminus T$ be given. If $x \in k$, then $x \in k/O_v^x$ (by the definition of $T$). An easy case by case analysis, namely $v(x) > 0$ or $v(x) < 0$, shows that $1 + x \in O_v^x \cup xO_v^x$, etc. Finally, if $x \notin k$, then letting $K_\alpha \subset K$ be the relative algebraic closure of $k(x)$ in $K$, one has: Since $x \in K \setminus T$, one must have $x \in K_\alpha \setminus T$. Thus by the discussion above, it follows that $1 + x \in T_\alpha \cup xT_\alpha$, and therefore, $1 + x \in T \cup xT$, etc.

This concludes the proof of Lemma 4.

\textbf{Step 2.} Using Lemma 4 above and applying the Arason–Elman–Jacob Theorem 3, we get: There exists a valuation $w$ on $K$ such that $|O_w^x/(O_w \cap T)| \leq 2$ and $1 + m_w \subset T$. Hence letting $O_w^x T \subset K^x$ be the subgroup generated by $T$ and $O_w$, one has $O_w/(O_w \cap T) = (O_w^x T)/T$, thus $|(O_w^x T)/T| \leq 2$.

We claim that one actually has $O_w^x \subset T$. Indeed, first, one has $k^x = O_v^x \cdot \pi^x$ as direct sum, hence $(k^x/p)/(O_v^x/p) = \pi^{x/p}$. Second, by definitions one has that $K^x/p = \Sigma/p \cdot k^x/p$ and $T/p = \Sigma/p \cdot O_v^x/p$, both of which being direct sums. Thus finally one gets that

$$K^x/p = \Sigma/p \cdot k^x/p = \Sigma/p \cdot O_v^x/p \cdot \pi^{x/p} = T/p \cdot \pi^{x/p}$$

where the dot denotes direct sums, in particular, one has $|K/T| = |(K^x/p)/(T/p)| = p$. Hence considering the canonical inclusions of groups $T \subset O_v^x, T \subset K^x$, conclude that the order $|(O_w^x T)/T| \leq 2$ must divide the order $|K^x/T| = p$. Since $2 < p$, we conclude that $|(O_w^x T)/T| = 1$, thus $T = O_w^x T$, hence $O_w^x \subset T$ is the only possibility. Since $|K^x/T| = p$, thus $K^x/O_w^x \geq p$, we conclude:

- The valuation $w$ is a non-trivial valuation of $K$. 

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Recalling that $\mathcal{O}_w^x \subset T$, one has that the canonical projection $K^x/\mathcal{O}_w^x \to K^x/T$ is surjective. Therefore, if $b \in K$ is a generator of $K^x/T$, e.g., $b = \pi \in k_\alpha$ has $v_\alpha(\pi) = 1$, then $b$ is not a $w$-unit and $w(b)$ is not divisible by $p$ in $wK = K^x/\mathcal{O}_w^x$, hence $wK$ is not divisible by $p$.

For every subfield $K_\alpha \subset K$ as in the proof of Lemma 4, let $v_\alpha := w|_{K_\alpha}$ be the restriction of $w$ to $K_\alpha$. Then $\mathcal{O}_{v_\alpha} = \mathcal{O}_w \cap K_\alpha$, and therefore, $\mathcal{O}_{v_\alpha}^x$ is contained in $T_\alpha = T \cap K_\alpha$.

**Lemma 5.** The restriction $v_\alpha := w|_{K_\alpha}$ of $w$ to $K_\alpha$ equals the $p$-adic valuation $w_\alpha$.

**Proof.** By the first part of the proof of Lemma 4, we have that $\mathcal{O}_{v_\alpha}^x \subset T_\alpha$. Since $\mathcal{O}_{v_\alpha}^x \subset T_\alpha$ as well, it follows that the element wise product $\mathcal{O}_{v_\alpha}^x \mathcal{O}_{w_\alpha}^x$ is contained in $T_\alpha$. Since $T_\alpha$ is a proper subgroup of $K_\alpha^x$, it follows that $\mathcal{O}_{v_\alpha}^x \mathcal{O}_{w_\alpha}^x \neq K_\alpha^x$ as well. The following is well known valuation theoretical non-sense: Let $n$ be the largest common ideal of $\mathcal{O}_{v_\alpha}$ and $\mathcal{O}_{w_\alpha}$. Then $\mathcal{O} := \mathcal{O}_{v_\alpha} \mathcal{O}_{w_\alpha}$ equals both the localization of $\mathcal{O}_{v_\alpha}$ at $n$ and the localization $\mathcal{O}_{w_\alpha}$ at $n$. Further, $\mathcal{O}$ is the smallest valuation ring of $K$ which contains both $\mathcal{O}_{v_\alpha}$ and $\mathcal{O}_{w_\alpha}$; or equivalently, $\mathcal{O}$ is the valuation ring of the finest common coarsening of $v_\alpha$ and $w_\alpha$. We now claim that one has:

$$\mathcal{O}^x = \mathcal{O}_{v_\alpha}^x \mathcal{O}_{w_\alpha}^x.$$  

Indeed, let $v_\alpha^1$ and $w_\alpha^1$ be the valuations of $\kappa(n) := \mathcal{O}/\mathcal{O}_n$ defined by $\mathcal{O}_{w_\alpha}/\mathcal{O}_n$, respectively $\mathcal{O}_{v_\alpha}/\mathcal{O}_n$. Then $v_\alpha^1$ and $w_\alpha^1$ are independent, one has exact sequences

$$1 \to (1 + n) \to \mathcal{O}_{v_\alpha}^x \to \mathcal{O}_{v_\alpha^1}^x \to 1 \quad \text{and} \quad 1 \to (1 + n) \to \mathcal{O}_{w_\alpha}^x \to \mathcal{O}_{w_\alpha^1}^x \to 1.$$ 

Since $v_\alpha^1$ and $w_\alpha^1$ are independent valuations of $\kappa(n)$, one has that $\mathcal{O}_{v_\alpha}^x \mathcal{O}_{w_\alpha}^x = (\kappa(n)^x)$, and therefore

$$(\mathcal{O}_{v_\alpha}^x \mathcal{O}_{w_\alpha}^x)/(1 + n) = (\kappa(n)^x).$$ 

On the other hand, one also has $\mathcal{O}^x/(1 + n) = (\kappa(n)^x)$. Further, $1 + n$ is contained in both $\mathcal{O}_{v_\alpha}^x$ and $\mathcal{O}_{w_\alpha}^x$, hence we conclude that $\mathcal{O}_{v_\alpha}^x \mathcal{O}_{w_\alpha}^x = \mathcal{O}^x$, as claimed.

By contradiction, suppose that $\mathcal{O}_{v_\alpha} \neq \mathcal{O}_{w_\alpha}$. Recall that the valuation ring $\mathcal{O}_{w_\alpha}$ has finite residue field, hence $\mathcal{O}_{w_\alpha}$ is minimal among the valuation rings of $K_\alpha$, and in particular, $\mathcal{O}_{v_\alpha}$ cannot be contained in $\mathcal{O}_{w_\alpha}$. Therefore, in the above notations, one has that $\mathcal{O}_{v_\alpha} \subset \mathcal{O}$ strictly, or equivalently, $\mathcal{O}_{v_\alpha} \subset \mathcal{O}_{w_\alpha}$ is a strict inclusion. On the other hand, if $b \in k$ is any element of minimal positive value $1_v$, then $m_{w_\alpha} = b \mathcal{O}_{w_\alpha}$, and therefore, $b \notin \mathcal{O}_{v_\alpha}$. Thus we have

$$b \in \mathcal{O}^x = \mathcal{O}_{v_\alpha}^x \mathcal{O}_{w_\alpha}^x \subset T_\alpha,$$

contradicting the fact that $w(b)$ generates $wK/w(T) \cong \mathbb{Z}/p$. Thus we conclude that one must have $\mathcal{O}_{w_\alpha} = \mathcal{O}_{v_\alpha}$, and Lemma 5 is proved.

We next claim that $w$ is a $p$-adic valuation of $K$ having $p$-adic rank $d_w = d_v$. Indeed, for $t \in \mathcal{O}_w$, let $K_\alpha \subset K$ be the relative algebraic closure of $k(t)$ in $K$. Then $K_\alpha|k$ has transcendence degree $\leq 1$, and therefore, $w|_{K_\alpha} = w_\alpha$ is the $p$-adic valuation $w_\alpha$ by Lemma 5. In particular, if $b \in k$ is such that $v(b) = 1_v$ is the minimal positive element of $v(K^x)$, it follows that $w_\alpha(b)$ is the minimal positive element of $w_\alpha K_\alpha$ under $vK \hookrightarrow w_\alpha K_\alpha$, and further $kv = K_\alpha w_\alpha$ is the finite field of cardinality $f_v = f_{w_\alpha}$. One has the following:

a) $w(b)$ is the minimal positive element of $w(K^x)$. Indeed, for $t \in m_w$, in the above notations one has: $w(t) = w_\alpha(t) \geq w_\alpha(b) = w(b)$.

b) $kv = K\ell$ thus $f_v = f_{w_\alpha}$. Indeed, if $t \in \mathcal{O}_w$, then in the above notations, the residue $\ell \in K\ell$ satisfies: $\ell \in K_\alpha w_\alpha = kv$.

Therefore, $w$ is a $p$-adic valuation of rank $d_w = d_v$, which is unique, by the uniqueness of $w_\alpha = w|_{K_\alpha}$ for every subfield $K_\alpha$. This concludes the proof of Theorem B.
4. Proof of the other announced results

A) Proof of Theorem A

The following stronger assertion holds (from which Theorem A immediately follows):

THEOREM 6. Let \( k/\mathbb{Q}_p \) be a finite extension containing the \( p^n \)th roots of unity, and let \( k_0 \subseteq k \) be a subfield which is relatively algebraically closed in \( k \). Let \( X_0 \) be a complete smooth \( k_0 \)-variety, and \( K_0 = k_0(X) \) be the function field of \( X_0 \). The following hold:

1) Every \( k \)-rational point \( x \in X_0 \) gives rise to a bouquet of conjugacy classes of liftable sections \( s'_x \) of \( \text{Gal}(K'_0|K_0) \rightarrow \text{Gal}(k'_0|k_0) \) above \( x \).

2) Suppose that \( p > 2 \), and let \( s' \) be a liftable section of \( \text{Gal}(K'_0|K_0) \rightarrow \text{Gal}(k'_0|k_0) \). Then there exists a unique \( k \)-rational point \( x \in X_0 \) such that \( s' \) equals one of the sections \( s'_x \) above.

Proof. The proof is very similar to the proof of Theorem A of Pop [P2]. We repeat here the arguments briefly for reader’s sake.

To 1): Let \( v \) be the valuation of \( k \). We notice that by Section 2, C), b), there exists a bijection from the set of (equivalence classes of) \( p \)-adic valuations \( w \) of \( K_0 = \kappa(X_0) \) with \( d_w = d_v \) onto the set of bouquets of liftable sections above \( k \)-rational points \( x \) of \( X_0 \), which sends each \( w \) to the corresponding bouquet of liftable sections above the center \( x \) of the canonical coarsening \( w_1 \) on \( X = X_0 \times_{k_0} k \). Conclude by applying assertion 1) of Theorem B.

To 2): Suppose that \( k_0 \subseteq k \) is relatively algebraically closed, it follows that \( k_0 \) is \( p \)-adically closed. Let \( v \) be the valuation of \( k \) and of all subfields of \( k \). Since \( k_0 \) is \( p \)-adically closed, we can apply Theorem B and get: For every liftable section \( s' \) of \( \text{Gal}(K'_0|K_0) \rightarrow \text{Gal}(k'_0|k_0) \), there exists a unique \( p \)-adic valuation \( w \) of \( K_0 \) which prolongs \( v \) to \( K_0 \) and has \( p \)-adic rank equal to the \( p \)-adic rank of \( v \), such that \( s' \) is a section above \( w \). Let \( w_1 \) be the canonical coarsening of \( w \). Then we have:

Case 1. The valuation \( w_1 \) is trivial.

Then \( w_1 \) is a discrete \( p \)-adic valuation of \( K \) prolonging \( v \) to \( K \), having the same residue field and the same value group as \( v \). Equivalently, the completions of \( k_0 \) and \( K_0 \) are equal, hence equal to \( k \). Therefore, \( w_1 \) is uniquely determined by the embedding \( \iota_w : (K_0, w) \rightarrow (k, v) \). In geometric terms, \( \iota_w \) defines a \( k \)-rational point \( x \) of \( X_0 \), etc.

Case 2. The valuation \( w_1 \) is not trivial.

Then \( w_1 \) is a \( k_0 \)-rational place of \( K_0 \), hence defines a \( k_0 \)-rational point \( x_0 \) of \( X_0 \); hence by base change, a \( k \)-rational point \( x \) of \( X_0 \) as well, etc. \( \Box \)

B) Proof of Theorem B°

The proof is almost identical with the one of Theorem B° from Pop [P2]. The proof of assertion 1) is identical with the proof of assertion 1) of Theorem B, thus we omit it. Concerning assertion 2), let \( s'_L : \text{Gal}(k'|l) \rightarrow \text{Gal}(K'|L) \) be a liftable section of \( \text{pr}'_L : \text{Gal}(K'|L) \rightarrow \text{Gal}(k'|l) \). Then the restriction of \( s'_L \) to \( \text{Gal}(k'|k) \subseteq \text{Gal}(k'|l) \) gives rise to a liftable section \( s' : \text{Gal}(k'|k) \rightarrow \text{Gal}(K'|K) \) of \( \text{pr}'_K : \text{Gal}(K'|K) \rightarrow \text{Gal}(k'|k) \). Hence by Theorem B, there exists a unique \( p \)-adic valuation \( w^1 \) of \( K \) which prolongs the \( p \)-adic valuation \( v_k \) of \( k \) to \( K \) and has \( d_{w^1} = d_{v_k} \), and \( s' = s_{w^1} \) in the usual way. Let \( w = w^1|_L \) be the restriction of \( w^1 \) to \( L \). Then \( w \) prolongs the valuation \( v \) of \( l \) to \( L \). We claim that \( w^1 \) is the unique prolongation of \( w \) to \( K \). Indeed, let \( w^2 := w \circ \sigma_0 \) with \( \sigma_0 \in \text{Gal}(k|l) \), be a further prolongation of \( w \) to \( K \). Then if \( (w^i)' \) is a prolongation of \( w^i \) to \( K' \), \( i = 1, 2 \), and \( \sigma \in \text{im}(s'_L) \) is a preimage of \( \sigma_0 \), then \( (w^2)' = (w^1)' \circ \sigma \) is a prolongation of \( w^2 \) to \( K' \). Therefore, if \( Z_{w^1} \subseteq \text{Gal}(K'|K) \) is the decomposition group above \( w^1 \), then \( Z_{w^1} \sigma = \sigma Z_{w^1} \sigma^{-1} \) is the decomposition group above \( w^2 \). On the other hand, \( \text{im}(s') \subseteq Z_{w^1} \) by Theorem B. Since \( \sigma \in \text{im}(s'_L) \), and \( \text{Gal}(k'|k) \) is a normal
subgroup of $\text{Gal}(k'|l)$, thus $\text{im}(s')$ is normal in $\text{im}(s'_L)$. Hence $\text{im}(s') \subseteq Z_w^1 \cap Z_w^2$. But then by the uniqueness assertion of Theorem B, we must have $w^1 = w^2$. Equivalently, $\text{im}(s'_L)$ is contained in $Z_w \subset \text{Gal}(K'|L)$. Finally conclude that $d_w = d_v$, as claimed, and this concludes the proof of Theorem B.

C) Proof of Theorem A

The following stronger assertion holds (from which Theorem A follows immediately):

**Theorem 7.** Let $l|\mathbb{Q}_p$ be a finite extension. Let $l_0 \subset l$ a relatively algebraically closed subfield, and $k_0|l_0$ a finite Galois extension with $\mu_p \subset k_0$. Let $Y_0$ be a complete smooth geometrically integral variety over $l_0$. Let $L_0 = \kappa(Y_0)$ the function field of $Y_0$, and $K_0 = L_0k_0$.

1) Every $l$-rational point $y \in Y_0$ gives rise to a bouquet of conjugacy classes of liftable sections $s'_y$ of $\text{Gal}(K'_0|L_0) \to \text{Gal}(k'_0|l_0)$ above $y$.

2) Let $p > 2$, and $s': \text{Gal}(k'_0|l_0) \to \text{Gal}(K'_0|L_0)$ be a liftable section of $\text{Gal}(K'_0|L_0) \to \text{Gal}(k'_0|l_0)$. Then there exists a unique $l$-rational point $y \in Y_0(l)$ such that $s'$ equals one of the sections $s'_y$ introduced at point 1) above.

**Proof.** The proof is identical with the proof of Theorem 4.1 above, with the only difference that one uses Theorem B, in stead of Theorem B.

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References


Birational Section Conjecture


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