THE BLOCK APPROXIMATION THEOREM*

by

Dan Haran

School of Mathematics, Tel Aviv University

Ramat Aviv, Tel Aviv 69978, Israel

e-mail: haran@post.tau.ac.il

and

Moshe Jarden

School of Mathematics, Tel Aviv University

Ramat Aviv, Tel Aviv 69978, Israel

e-mail: jarden@post.tau.ac.il

and

Florian Pop

Department of Mathematics, University of Pennsylvania

Philadelphia, PA 19104-6395, USA

e-mail: pop@math.upenn.edu

Abstract: We prove the following theorem.

The P-adic Block Approximation Theorem: *Let $\mathcal{F}$ be a finite set of* P*-adic fields closed under Galois isomorphism and K a $P\mathcal{F}C$ field. Set $\mathcal{X} = \mathrm{AlgExt}(K, \mathcal{F})_{\min}$. For each $F \in \mathcal{X}$ let v_F be the unique* P*-adic valuation of F. Let I_0 be a finite set. For each $i \in I_0$ let $\mathcal{X}_i$ be an étale open-closed subset of $\mathcal{X}$, L_i a finite separable extension of K contained in K_s, and $c_i \in K^\times$. Suppose $\mathcal{X} = \bigcup_{i\in I_0} \bigcup_{\sigma\in \mathrm{Gal}(K)} \mathcal{X}_i^\sigma$. Suppose further, for all $i \in I_0$ and all $\sigma \in \mathrm{Gal}(K)$ we have $\mathcal{X}_i^\sigma = \mathcal{X}_j$ if and only if $i = j$ and $\sigma \in \mathrm{Gal}(L_i)$; otherwise $\mathcal{X}_i^\sigma \cap \mathcal{X}_i = \emptyset$. Assume $L_i \subseteq K_v$ for each $K_v \in \mathcal{X}_i$. Let V be an affine absolutely irreducible variety defined over K. For each $i \in I_0$ let $\mathbf{a}_i \in V_{\mathrm{simp}}(L_i)$. Then there exists $\mathbf{a} \in V(K)$ such that $v_F(\mathbf{a} - \mathbf{a}_i) > v_F(c_i)$ for each $i \in I_0$ and for every $F \in \mathcal{X}_i$.*

MR Classification: 12E30

Directory: \Jarden\Diary\HJPc

---

* Research supported by the Minkowski Center for Geometry at Tel Aviv University, established by the Minerva Foundation, by an ISF grant, and by the European Community Marie Curie Research Training Network GTEM, by NSF research grant DMS 0801144, and by the John Templeton Foundation grant ID 13394.

Introduction

The block approximation property of a field K is a local-global principle for absolutely irreducible varieties defined over K on the one hand and a weak approximation theorem for valuations and orderings on the other hand. It was proved in [FHV94] for orderings. Moreover, [HJP07] constructs fields with the block approximation property for valuations. In this work we prove the block approximation property for valuations in the most general settings.

More technically, the block approximation property of a proper field-valuation structure $\mathbf{K} = (K, X, K_x, v_x)_{x\in X}$ is a quantitative local-global principle for absolute irreducible varieties over K. Here K is a field and X is a profinite space on which the absolute Galois group $\mathrm{Gal}(K)$ of K continuously acts. Each K_x is a separable algebraic extension of K equipped with a valuation v_x. Given an absolutely irreducible variety V over K, open-closed subsets $X_1, \ldots, X_n$ of X (called **blocks**), and points $\mathbf{a}_1, \ldots, \mathbf{a}_n \in V_{\mathrm{simp}}(K_s)$ satisfying certain compatibility conditions, the block approximation property gives an $\mathbf{a} \in V(K)$ which is v_x-close to $\mathbf{a}_i$ for $i = 1, \ldots, n$ and every $x \in X_i$.

The block approximation property of $\mathbf{K}$ has several far reaching consequences: For each $x \in X$, $\mathrm{Aut}(K_x/K) = 1$ and the valued field (K_x, v_x) is the Henselian closure of $(K, v_x|_K)$. If $x_1, \ldots, x_n$ are non-conjugate elements of X, then $v_{x_1}|_K, \ldots, v_{x_n}|_K$ satisfy the weak approximation theorem. Finally, K is P$\mathcal{X}$C, where $\mathcal{X} = \{K_x \mid x \in X\}$. This means that every absolutely irreducible variety V with a simple K_x-rational point for each $x \in X$ has a K-rational point [HJP07, Prop. 12.3].

The main result of [HJP07] characterizes proper projective group structures as absolute Galois group structures of proper field-valuation structures having the block approximation property. In particular, the local fields of the field-valuation structures turns out to be Henselian closures of K.

In [HJP05] we replace the general Henselian fields of [HJP07] by P-adically closed fields. Here we call a field F P**-adically closed** if it is elementarily equivalent to a finite extension of $\mathbb{Q}_p$ for some p. The main result of [HJP05] considers a finite set $\mathcal{F}$ of P-adically closed fields **closed under Galois isomorphism** (i.e. if $\mathbb{F}, \mathbb{F}'$ are P-adically closed fields such that $\mathrm{Gal}(\mathbb{F}) \cong \mathrm{Gal}(\mathbb{F}')$ and $\mathbb{F} \in \mathcal{F}$, then $\mathbb{F}' \in \mathcal{F}$.) It says that G is

isomorphic to the absolute Galois group of a P$\mathcal{F}$C field K if and only if G is $\mathcal{F}$-projective and $\mathrm{Subgr}(G,\mathbb{F})$ is strictly closed in $\mathrm{Subgr}(G)$ for each $\mathbb{F}\in\mathcal{F}$.

The condition "G is $\mathcal{F}$-projective" is very mild. It says, every finite embedding problem for G which is $\mathcal{F}$-locally solvable is globally solvable. Nevertheless, adding the second condition that $\mathrm{Subgr}(G,\mathbb{F})$ is strictly closed in $\mathrm{Subgr}(G)$ for each $\mathbb{F}\in\mathcal{F}$, the group G can be extended to a proper projective group structure $\mathbf{G}$ [HJP05, Thm. 10.4]. Then the main theorem of [HJP07] is applied to realize $\mathbf{G}$ as the absolute Galois structure of a field-valuation structure $\mathbf{K}=(K,X,K_x,v_x)$ having the block approximation property [HJP05, Thm. 11.3]. In particular, K is then P$\mathcal{F}$C, that is K is P$\mathcal{X}$C, where $\mathcal{X}=\mathrm{AlgExt}(K,\mathcal{F})_{\min}$. The latter symbol stands for the set of minimal fields in the set $\mathrm{AlgExt}(K,\mathcal{F})=\bigcup_{\mathbb{F}\in\mathcal{F}}\mathrm{AlgExt}(K,\mathbb{F})$, where $\mathrm{AlgExt}(K,\mathbb{F})$ is the set of all algebraic extensions of K that are elementarily equivalent to $\mathbb{F}$.

Conversely, let K be a P$\mathcal{F}$C field. Then $\mathrm{AlgExt}(K,\mathbb{F})$ is strictly closed in $\mathrm{AlgExt}(K)$ [HJP05, Lemma 10.1] and $\mathrm{Gal}(K)$ is $\mathcal{F}$-projective [HJP05, Prop. 4.1]. It follows from the preceding paragraph that $\mathrm{Gal}(K)\cong\mathrm{Gal}(K')$ for some field extension K' of K that admits a field-valuation structure having the block approximation property.

The goal of the present work is to prove that if K is P$\mathcal{F}$C, then one may choose $K'=K$ in the preceding theorem. In other words, the natural field-valuation structure $\mathbf{K}_{\mathcal{F}}$ attached to K and $\mathcal{F}$ has the block approximation property. For the convenience of the reader we reformulate this result without referring to field-valuation structures.

The P-adic Block Approximation Theorem: *Let $\mathcal{F}$ be a finite set of* P*-adic fields closed under Galois isomorphism and K a P$\mathcal{F}$C field. Set $\mathcal{X}=\mathrm{AlgExt}(K,\mathcal{F})_{\min}$. For each $F\in\mathcal{X}$ let v_F be the unique* P*-adic valuation of F. Let I_0 be a finite set. For each $i\in I_0$ let $\mathcal{X}_i$ be an étale open-closed subset of $\mathcal{X}$, L_i a finite separable extension of K contained in K_s, and $c_i\in K^\times$. Suppose $\mathcal{X}=\bigcup_{i\in I_0}\bigcup_{\sigma\in\mathrm{Gal}(K)}\mathcal{X}_i^\sigma$. Suppose further, for all $i\in I_0$ and all $\sigma\in\mathrm{Gal}(K)$ we have $\mathcal{X}_i^\sigma=\mathcal{X}_j$ if and only if $i=j$ and $\sigma\in\mathrm{Gal}(L_i)$; otherwise $\mathcal{X}_i^\sigma\cap\mathcal{X}_i=\emptyset$. Assume $L_i\subseteq K_v$ for each $K_v\in\mathcal{X}_i$. Let V be an affine absolutely irreducible variety defined over K. For each $i\in I_0$ let $\mathbf{a}_i\in V_{\mathrm{simp}}(L_i)$. Then there exists $\mathbf{a}\in V(K)$ such that $v_F(\mathbf{a}-\mathbf{a}_i)>v_F(c_i)$ for each $i\in I_0$ and for every $F\in\mathcal{X}_i$.*

A block approximation theorem for real closed fields is proved in [FHV94, Prop. 1.2]: Let K be a PRC field, $\mathcal{X}$ a strictly closed system of representatives for the $\mathrm{Gal}(K)$-orbits of the real closures of K, and $\mathcal{X} = \dot{\bigcup}_{i\in I} \mathcal{X}_i$ a partition of $\mathcal{X}$ with $\mathcal{X}_i$ open-closed. Let V be an absolutely irreducible variety defined over K. For each $i \in I$ let $\mathbf{a}_i$ be a simple point of V contained in $V(F)$ for each $F \in \mathcal{X}_i$. Then there exists $\mathbf{a} \in V(K)$ which is F-close to $\mathbf{a}_i$ for each $i \in I$ and each $F \in \mathcal{X}_i$.

The easy proof of the real block approximation theorem takes advantage of the function X^2 whose values are totally positive and of the assumption on $\mathcal{X}$ being a strictly closed system of representatives of the real closures of K. The assumption on the existence of a strictly closed system of representatives holds for every field K [HaJ85, Cor. 9.2].

In the P-adic case we can prove a similar result about systems of representatives only in some cases, e.g. if $\mathcal{F} = \{\mathbb{Q}_p\}$ or if K is countable. But we do not know whether in the general case the $\mathrm{Gal}(K)$-orbits of $\mathrm{AlgExt}(K, \mathcal{F})_{\min}$ have a closed (in the étale topology) system of representatives. Fortunately, the conditions on the blocks in the P-adic block approximation theorem can be always realized and they turn out to be sufficient for the proof of the block approximation theorem.

The P-adic block approximation theorem is based on a block approximation theorem for field-valuation structures $\mathbf{K} = (K, X, K_x, v_x)_{x\in X}$ with bounded residue fields (Theorem 4.1). Instead of the function X^2 used in the proof of the block approximation theorem for real closed fields our proof uses a function $\wp(X)$ with good P-adic properties. In particular, its values are totally P-adically integral (Section 3). We also use that if K is P$\mathcal{X}$C, with $\mathcal{X} = \{K_x \mid x \in X\}$, then K is v_x-dense in K_x for each $x \in X$.

The next step is to extend the P$\mathcal{F}$C field K of the P-adic block approximation theorem to a field-valuation structure $\mathbf{K} = (K, X, K_x, v_x)_{x\in X}$ such that $\mathcal{X} = \mathrm{AlgExt}(K, \mathcal{F})_{\min}$. There are two essential points in the proof. First we prove that $\mathrm{Aut}(K_x/K) = 1$ for each $x \in X$ (essentially Proposition 2.3(b)). Then we prove that for each finite extension L of K the map of $\mathcal{X}_L = \{K_x \in \mathcal{X} \mid L \subseteq K_x\}$ into $\mathrm{Val}(L)$ given by $v_x \mapsto v_x|_L$ is étale continuous (Lemma 5.12].

1. On the Algebraic Topological Closure of a Valued Field

The completion of a rank one valued field (K, v) is the ring of all Cauchy sequences modulo 0-sequences. A similar construction works for an arbitrary valued field (K, v). While the algebraic part in the rank one case is both the Henselization K_v of (K, v) and the v-closure of K in K_s, the v-closure $K_{v,\mathrm{alg}}$ of K in K_s in the general case is only contained in K_v. Nevertheless, as we shall see below, $K_{v,\mathrm{alg}}$ shares several properties with K_v.

Definition 1.1: Cauchy sequences. Let v be a valuation of a field K. We denote the valuation ring of v by O_v and its residue field by $\bar{K}_v$. Unless we say otherwise, we assume that v is **nontrivial**; that is O_v is a proper subring of K. Occasionally we also speak about the **trivial valuation** v_0 of K with $O_{v_0} = K$.

Let λ be a limit ordinal. A **sequence** (of **length** λ of elements of K) is a function x from the set of all ordinals smaller than λ (usually one identifies this set with λ itself) to K. We denote the value of x at $\kappa < \lambda$ by x_κ and the whole sequence by $(x_\kappa)_{\kappa<\lambda}$. The sequence $(x_\kappa)_{\kappa<\lambda}$ **converges** to an element a of K if for each $c \in K^\times$ there is a $\kappa_0 < \lambda$ with $v(x_\kappa - a) > v(c)$ for all $\kappa \geq \kappa_0$. We say $(x_\kappa)_{\kappa<\lambda}$ is a **Cauchy sequence** if for each $c \in K^\times$ there is a κ_0 with $v(x_\kappa - x_{\kappa'}) > v(c)$ for all $\kappa, \kappa' \geq \kappa_0$. Finally, (K, v) is **complete** if every Cauchy sequence in K converges. ∎

Proposition 1.2 ([Ax71, p. 173, Prop. 8]): *Every valued field (K, v) has an extension $(\hat{K}, \hat{v})$ which is complete such that (K, v) is dense in $(\hat{K}_v, \hat{v})$. This extension is unique up to a K-isomorphism.*

Remark 1.3: The valuation ring of $\hat{K}_v$. Denote the valuation ring of $\hat{K}_v$ by $\hat{O}_v$. It is the closure of O_v in $\hat{K}_v$ under the $\hat{v}$-topology. In analogy to the presentation of $\mathbb{Z}_p$ as an inverse limit $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$, we present $\hat{O}_v$ as an inverse limit of quotient rings of O_v.

To this end let Γ_v be the value group of (K, v). For each nonnegative $\alpha \in \Gamma_v$ consider the ideal $\mathfrak{m}_\alpha = \{a \in K \mid v(a) > \alpha\}$ of O_v. We prove that there is a natural isomorphism $\varprojlim O_v/\mathfrak{m}_\alpha \cong \hat{O}_v$.

Choose a well ordered cofinal subset Δ of Γ_v. For each $x = (x_\alpha + \mathfrak{m}_\alpha)_{\alpha\in\Gamma_v}$ in $\varprojlim O_v/\mathfrak{m}_\alpha$ the sequence $(x_\alpha)_{\alpha\in\Delta}$ is Cauchy. Hence, it converges to an element $\hat{x}$ of $\hat{O}_v$

which is independent of the representatives x_α of $x_\alpha + \mathfrak{m}_\alpha$.

Conversely, let $\hat{x} \in \hat{O}_v$. For each $\alpha \in \Gamma$ choose $x_\alpha \in O_v$ with $\hat{v}(x_\alpha - \hat{x}) > \alpha$. If $\beta > \alpha$, then $v(x_\beta - x_\alpha) > \alpha$. So, $x_\beta \equiv x_\alpha \mod \mathfrak{m}_\alpha$. This gives a well defined element $x = (x_\alpha + \mathfrak{m}_\alpha)_{\alpha \in \Gamma_v}$ of $\varprojlim O_v/\mathfrak{m}_\alpha$ which is mapped to $\hat{x}$ under the map of the preceding paragraph.

The map $x \mapsto \hat{x}$ is the promised isomorphism. ∎

Notation 1.4: We denote the set of all valuations of K and the trivial valuation by $\mathrm{Val}(K)$.

Let $v, w \in \mathrm{Val}(K)$. We say v is **finer** than v' (and v' is **coarser** than v) and write $v \prec v'$ if $O_v \subseteq O_{v'}$. In particular, the trivial valuation is coarser than every $v \in \mathrm{Val}(K)$. ∎

Remark 1.5: Dependent valuations. Valuations v and v' of K are **dependent** if K has a valuation v'' which is coarser than both v and v'; equivalently, if the ring $O_v O_{v'} = \{\sum_{i=1}^n a_i a_i' \mid a_i \in O_v,\ a_i' \in O_{v'}\}$ is a proper subring of K. This is the case if and only if the v-topology of K coincides with the v'-topology [Jar91b, Lemma 3.2(a) and Lemma 4.1]. Denote the common topology by T.

The definitions of a Cauchy sequence and the convergence of transfinite sequences, as well of the concept of density can be rephrased in terms of the T-topology. For example, a sequence $(x_\kappa)_{\kappa<\lambda}$ is Cauchy if and only if the following holds: For every T-open neighborhood U of 0 in K there is an κ_0 with $x_\kappa - x_{\kappa'} \in U$ for all $\kappa, \kappa' \geq \alpha_0$. Hence, $(\hat{K}_v, \hat{v})$ depends only on the topology T. Thus, $\hat{K}_v$ can be identified with $\hat{K}_{v'}$. If $O_v \subseteq O_{v'}$, than $\hat{O}_v \subseteq \hat{O}_{v'}$, because $\hat{O}_v$ is the T-closure of O_v and $\hat{O}_{v'}$ is the T-closure of $O_{v'}$ in $\hat{K}_v$. ∎

Remark 1.6: The separable algebraic part of the completion. Let $(\hat{K}_v, \hat{v})$ be the completion of a valued field (K, v). In general, $\hat{K}_v$ is not a separable extension of K. Nevertheless, we denote the maximal valued subfield of $(\hat{K}_v, \hat{v})$ which is separable algebraic over K by $(K_{v,\mathrm{alg}}, v_{\mathrm{alg}})$. Since $(\hat{K}_v, \hat{v})$ is unique up to a K-isomorphism, so is $(K_{v,\mathrm{alg}}, v_{\mathrm{alg}})$.

We extend $\hat{v}$ to a valuation $\hat{v}_s$ of the separable closure $(\hat{K}_v)_s$ of $\hat{K}_v$ and embed

K_s in $(\hat{K}_v)_s$. Let v_s be the restriction of $\hat{v}_s$ to K_s. Then $K_{v,\mathrm{alg}}$ is the closure of K in K_s under the v_s-topology. Thus, a choice of $\hat{v}_s$ and an embedding of K_s in $(\hat{K}_v)_s$ determines $K_{v,\mathrm{alg}}$ uniquely within K_s.

Suppose w is a valuation coarser than v. Then $\hat{K}_w = \hat{K}_v$ (Remark 1.5). By the first paragraph of this remark, $K_{w,\mathrm{alg}} = K_{v,\mathrm{alg}}$.

Let $D_{v_s} = \{\sigma \in \mathrm{Gal}(K) \mid \sigma O_{v_s} = O_{v_s}\}$ be the **decomposition group** of v_s over K. Let K_v be the fixed field of D_{v_s} in K_s and v_h be the restriction of v_s to K_v. Then (K_v, v_h) is the Henselian closure of (K, v). It is determined by v up to a K-isomorphism.

Each $\sigma \in D_{v_s}$ preserves the v_s-topology of K_s, hence fixes each element of the v_s-closure of K in K_s, so $\sigma \in \mathrm{Gal}(K_{v,\mathrm{alg}})$. Therefore, $K_{\mathrm{alg},v} \subseteq K_v$.

Suppose $\mathrm{rank}(v) = 1$; that is, no nontrivial valuation of K is strictly coarser than v. Alternatively, the value group of v is isomorphic to a subgroup of $\mathbb{R}$ [Jar91b, Lemma 3.4]. Then $\hat{K}_v$ is Henselian (Hensel's lemma). Hence, $K_{v,\mathrm{alg}} = K_s \cap \hat{K}_v$ is also Henselian. It follows that $K_{v,\mathrm{alg}} = K_v$. ∎

Remark 1.7: Definition of $\bigcap_{w \succ v} K_w$. We consider a $v \in \mathrm{Val}(K)$. If v is a valuation, we extend v to a valuation v_s of K_s, otherwise we let v_s be the trivial valuation of K_s. Then we extend each $w \in \mathrm{Val}(K)$ which is coarser than v to a valuation w_s which is coarser than v_s [Jar91b, Lemma 9.4]. The map $w \mapsto w_s$ is a bijection of the set of all valuations of K coarser than v onto the set of all valuations of K coarser than v_s. Its inverse is the map $w_s \mapsto w_s|_K$. Moreover, if $w \prec w'$, then $w_s \prec w'_s$. Indeed, since both w_s and w'_s are coarser than v_s, they are comparable [Jar91b, Lemma 3.2]. Hence, $w_s \prec w'_s$ or $w'_s \prec w_s$. In the latter case, $w = w'$, so by [Jar91b, Cor. 6.6], $w_s = w'_s$.

Let $D(w_s)$ be the decomposition group of w_s over K. We denote the fixed field of $D(w_s)$ in K_s by K_w and put $w_h = w_s|_{K_w}$. Then (K_w, w_h) is a Henselian closure of (K, w). If $w \prec w'$, then $D_{w_s} \subseteq D_{w'_s}$ and $K_{w'} \subseteq K_w$ [Jar91b, Prop. 9.5].

It follows that $\bigcap_{w \succ v} K_w$ is an extension of K which is well defined up to a K-isomorphism. Since $K_{v,\mathrm{alg}} = K_{w,\mathrm{alg}}$ for each $w \succ v$ [Remark 1.6], $K_{v,\mathrm{alg}} \subseteq \bigcap_{w \succ v} K_w \subseteq K_v$. ∎

Lemma 1.8 ([Eng78, Thm. 2.11]): *Let (K, v) be a valued field. For each valuation w*

of K which is coarser than v we choose a Henselian closure (K_w, w_h) with $K_w \subseteq K_v$ and $v_h|_{K_w} = w_h$. Then $K_{v,\mathrm{alg}} = \bigcap_{w \succ v} K_w$.

Proof: By Remark 1.7, $L = \bigcap_{w \succ v} K_w$ is well defined. Moreover, the set of all valuations w of K which are coarser than v is linearly ordered. That is, if $v \prec w, w'$, than either $O_w \subseteq O_{w'}$ or $O_{w'} \subseteq O_w$ [Jar91b, Lemma 3.2]. Hence, $O = \bigcup_{w \succ v} O_w$ is either a valuation ring of K or K itself.

Case A: *O is the valuation ring of a valuation w_0. (We say that v is* **bounded***).* In this case w_0 is finer than no other valuation of K. Hence, $\mathrm{rank}(w_0) = 1$. Therefore, $L = K_{w_0} = K_{w_0,\mathrm{alg}} = K_{v,\mathrm{alg}}$ (Remark 1.6).

Case B: *$O = K$ (We say that v is* **unbounded***).* It suffices to prove K is v-dense in L. Consider $w, w' \succ v$. Denote the restriction of w_h (resp. $(w')_h$, v) to L by w_L (resp. w'_L, v_L). By our choice $w \prec w'$ if and only if $w_L \prec w'_L$. Hence, v_L is unbounded. Therefore, $\bigcup_{w \succ v} O_{w_L} = L$ and $\bigcap_{w \succ v} \mathfrak{m}_{w_L} = 0$.

Let now $x, c \in L^\times$. Then, there is $w \succ v$ with $c, c^{-1}, x, x^{-1} \notin \mathfrak{m}_{w_L}$. Thus, $w_L(c) = w_L(x) = 0$. Since $K \subseteq L \subseteq K_w$, the residue field of L at w_L is $\bar{K}_w$. Hence, there is $a \in K$ with $w_L(a - x) > 0 = w_L(c)$. Since $\mathfrak{m}_{w_L} \subseteq \mathfrak{m}_{v_L}$, this gives $v_L(a - x) > v_L(c)$, as desired. ∎

Lemma 1.9: *Let $f \in K_{v,\mathrm{alg}}[X]$. Suppose f has a zero in K_w for each $w \succ v$. Then f has a zero in $K_{v,\mathrm{alg}}$.*

Proof: Let $x_1, \ldots, x_n$ be the zeros of f in $\tilde{K}$. Assume $x_1, \ldots, x_n \notin K_{v,\mathrm{alg}}$. Then, for each i there is $w_i \succ v$ with $x_i \notin K_{w_i}$ (Lemma 1.8). Let w be the coarsest valuation among $w_1, \ldots, w_n$. Thus, $K_w \subseteq K_{w_i}$, $i = 1, \ldots, n$, so $x_1, \ldots, x_n \notin K_w$. In other words, f has no zero in K_w, a contradiction. ∎

The following result generalizes a lemma of Kaplansky-Krasner [FrJ86, Lemma 10.13]. Its proof is included in the proof of [Pop90, Lemma 2.7].

Lemma 1.10: *Let $f \in K_{v,\mathrm{alg}}[X]$. Suppose f has no root in $K_{v,\mathrm{alg}}$. Then f is bounded away from 0. That is, 0 has a v-open neighborhood U in $K_{v,\mathrm{alg}}$ such that $f(K_{v,\mathrm{alg}}) \cap U = \emptyset$.*

Proof: Lemma 1.9 gives $w \succ v$ with no roots in K_w of f. Kaplansky-Krasner for Henselian fields [FrJ, Lemma 10.13] gives a w-open neighborhood U_w of 0 in K_w with $f(K_w) \cap U_w = \emptyset$. Then $U = K_{v,\mathrm{alg}} \cap U_w$ is a w-open, hence also v-open, neighborhood of 0 in $K_{v,\mathrm{alg}}$ which satisfies $f(K_{v,\mathrm{alg}}) \cap U = \emptyset$. ∎

PROPOSITION 1.11: *Consider valuations v and w of K. Suppose $K_v = K_{v,\mathrm{alg}}$, $K_w = K_{w,\mathrm{alg}}$, and $K_v \neq K_w$. Then $K_v K_w = K_s$.*

Proof: Put $M = K_v K_w$. Assume $M \neq K_s$. With the notation of Remark 1.6, let v_s (resp. v_M) be the restriction of $\hat{v}_s$ to K_s (resp. M). Define w_M and w_s analogously. Then M is Henselian with respect to both v_M and w_M. Hence, v_M and w_M are dependent [Jar91b, Lemma 13.2]. Therefore, they define the same topology T on M.

By Remark 1.6, K_v is the closure of K in K_s in the v_s-topology, so K_v is the T-closure of K in M. Similarly K_w is the T-closure of K in M. It follows, $K_v = K_w$, in contradiction to our assumption. ∎

Definition 1.12: The core of a valuation. Let (K, v) be a valued field. Suppose $\bar{K}_v$ is separably closed. Denote the set of all $w \in \mathrm{Val}(K)$ with $v \prec w$ and $\bar{K}_w$ separably closed by $V(v)$. If $w \in V(v)$, $w_0 \in \mathrm{Val}(K)$ and $v \prec w_0 \prec w$, then $\bar{K}_{w_0}$ is a residue field of $\bar{K}_w$. Hence, $\bar{K}_{w_0}$ is separably closed and $w_0 \in V(v)$.

Let $O = \bigcup_{w \in V(v)} O_w$. The right hand side is an ascending union of overrings of O_v (i.e. subrings of K containing O_v). Hence, O is an overring of O_v. As such, either O is a valuation ring of K or K itself. Let v_{core} be the corresponding valuation in the former case and the trivial valuation in the latter case. Call v_{core} the **core** of v.

To make the definition complete, we define $V(v) = \{v\}$ and $v_{\mathrm{core}} = v$ if $\bar{K}_v$ is not separably closed. This definition follows the convention of [Pop88] but is slightly different from that of [Pop94]. ∎

Let K be a non-separably-closed field and v_1, v_2 Henselian valuations of K. If $\bar{K}_{v_1}$ or $\bar{K}_{v_2}$ are not separably closed, then by F.K.Schmidt-Engler v_1 and v_2 are comparable [Jar91b, Prop 13.4]. The following result completes this statement in the case where both $\bar{K}_{v_1}$ and $\bar{K}_{v_2}$ are separably closed. It appears without proof as [Pop88, Satz 1.9].

Lemma 1.13 ([Pop94, Prop. 1.3]): *Let v_1 and v_2 be Henselian valuations of a field K. Suppose K is not separably closed. Then $v_{1,\mathrm{core}}$ and $v_{2,\mathrm{core}}$ are comparable.*

Proof: We consider two cases.

Case A: *v_1 and v_2 are comparable.* Without loss we may assume that $v_1 \prec v_2$. Then $\bar{K}_{v_1}$ is a residue field of $\bar{K}_{v_2}$. We first suppose $\bar{K}_{v_1}$ is not separably closed. Then $\bar{K}_{v_2}$ is not separably closed. So, $v_{1,\mathrm{core}} = v_1$ and $v_{2,\mathrm{core}} = v_2$. Therefore, $v_{1,\mathrm{core}} \prec v_{2,\mathrm{core}}$.

Next we suppose $\bar{K}_{v_1}$ is separably closed but $\bar{K}_{v_2}$ is not. Let $w \in V(v_1)$. Then $v_1 \prec w$ and $\bar{K}_w$ is separably closed. Hence, w is not coarser than v_2, so w must be finer than v_2. It follows, $v_{1,\mathrm{core}} \prec v_2 = v_{2,\mathrm{core}}$.

Finally we suppose both $\bar{K}_{v_1}$ and $\bar{K}_{v_2}$ are separably closed. Then $v_2 \in V(v_1)$ and $v_{1,\mathrm{core}} = v_{2,\mathrm{core}}$.

Case B: *v_1 and v_2 are incomparable.* By assumption, K is not separably closed. Hence, both $\bar{K}_{v_1}$ and $\bar{K}_{v_2}$ are separably closed. Moreover, K has a valuation w with $v_1, v_2 \prec w$ and $\bar{K}_w$ separably closed [Jar91b, Proposition 13.4]. Hence, by the third paragraph of Case A, $v_{1,\mathrm{core}} = w_{\mathrm{core}} = v_{2,\mathrm{core}}$. ∎

We use the notion of the core of a valuation to supplement a result of F.K.Schmidt-Engler saying that if $\bar{K}_v$ is not separably closed, then $\mathrm{Aut}(K_v/K) = 1$ [Jar91b, Prop. 14.5].

Proposition 1.14: *Let (K, v) be a valued field. Suppose $K_v = K_{v,\mathrm{alg}}$ but $K_v \neq K_s$. Then $\mathrm{Aut}(K_v/K)$ is trivial.*

Proof: Consider $\sigma \in \mathrm{Aut}(K_v/K)$. Let K' be the fixed field of σ in K_v and v' the restriction of v_h to K'. Then $K_v = K'_{v'}$. Also, K_v is the v_s-closure of K' in K_s (Remark 1.6). Hence, $K_v = K'_{v',\mathrm{alg}}$. Replace therefore (K, v) by (K', v'), if necessary, to assume K_v/K is Galois.

The field K_v is Henselian with respect to both v_h and $v_h \circ \sigma$. By Lemma 1.13, $(v_h)_{\mathrm{core}}$ and $(v_h \circ \sigma)_{\mathrm{core}}$ are comparable. Since $(v_h \circ \sigma)_{\mathrm{core}} = (v_h)_{\mathrm{core}} \circ \sigma$, the valuations $(v_h)_{\mathrm{core}}$ and $(v_h)_{\mathrm{core}} \circ \sigma$ are comparable. In addition, $(v_h)_{\mathrm{core}}|_K = (v_h)_{\mathrm{core}} \circ \sigma|_K$. Hence,

by [Jar91b, Cor. 6.6], $(v_h)_{\mathrm{core}} = (v_h)_{\mathrm{core}} \circ \sigma$. Thus, σ belongs to the decomposition group D of $(v_h)_{\mathrm{core}}$ in $\mathrm{Gal}(K_v/K)$.

Denote the restriction of $(v_h)_{\mathrm{core}}$ to K by w. Then $v \prec w$. Hence, $K_v = K_{v,\mathrm{alg}} \subseteq K_w \subseteq K_v$. So, $K_w = K_v$. This implies D is trivial. It follows from the preceding paragraph that $\sigma = 1$. ∎

2. Henselian Closures of P$\mathcal{X}$C Fields

A valued field (K, v) is v-dense in $K_{v,\mathrm{alg}}$ but not necessarily in its Henselization. However, under favorable conditions, this is the case.

We consider a field K, a fixed separable closure K_s of K, and denote the family of all extensions of K in K_s by $\mathrm{SepAlgExt}(K)$. A basis for the **étale topology** of $\mathrm{SepAlgExt}(K)$ is the collection of all sets $\mathrm{SepAlgExt}(L)$, where L is a finite separable extension of K [HJP07, Section 1].

Let $\mathcal{X}$ be an étale compact subset of $\mathrm{SepAlgExt}(K)$, K' a minimal field in $\mathcal{X}$, and v a valuation of K'. Suppose K is P$\mathcal{X}$C and (K', v) is Henselian. We prove that K is v-dense in K' and (K', v) is a Henselian closure of $(K, v|_K)$ (Proposition 2.3). An analogous result holds when v is replaced by an ordering.

We recall here that K is said to be P$\mathcal{X}$C, if each absolutely irreducible variety over K with a simple K'-rational point for each $K' \in \mathcal{X}$ has a K-rational point.

Lemma 2.1: *Let F be a separable algebraic extension of K and M an arbitrary extension of K. Suppose every irreducible polynomial $f \in K[X]$ with a root in F has a root in M. Then there is a K-embedding of F in M.*

Proof: For each finite extension L of K in F let $\mathrm{Embd}_K(L, M)$ be the set of all K-embedding of L in M. It is a nonempty finite set. Indeed, let x be a primitive element for L/K and $f = \mathrm{irr}(x, K)$. By assumption, f has a root x' in M. The map $x \mapsto x'$ extends to a K-embedding of L into M.

Suppose L' is a finite extension of L in F. Then the restriction from L' to L maps $\mathrm{Embd}_K(L', M)$ into $\mathrm{Embd}_K(L, M)$. The inverse limit of all $\mathrm{Embd}_K(L, M)$ is nonempty. Each element in the inverse limit defines a K-embedding of F into M. ∎

The following result is an elaboration of [Pop90, Lemma 2.7].

Lemma 2.2: *Let $\mathcal{X}$ be an étale compact subset of* $\mathrm{SepAlgExt}(K)$ *and v a valuation of K. Suppose K is P$\mathcal{X}$C. Then the following holds.*

(a) *Let $f \in K[X]$ be a separable polynomial. Suppose f has a zero in each $K' \in \mathcal{X}$. Then f has a zero in $K_{v,\mathrm{alg}}$.*

(b) *There is a $K' \in \mathcal{X}$ that can be K-embedded in $K_{v,\mathrm{alg}}$.*

Proof of (a): Assume f has no root in $K_{v,\mathrm{alg}}$. Choose $b \in K$ with $f'(b) \neq 0$ and let $c = f(b)$. Then $c \neq 0$. Lemma 1.10 gives a v-open neighborhood U of 0 in $K_{v,\mathrm{alg}}$ with

$$f(K_{v,\mathrm{alg}}) \cap U = \emptyset \tag{2}$$

Choose $d \in K^\times$ with

$$\{y \in K_{v,\mathrm{alg}} \mid v(y) > v(d)\} \subseteq U. \tag{3}$$

Finally choose $e \in K^\times$ with $v(e) > 2v(d) - v(c)$.

Now consider $x \in K$. By (2), $f(x) \notin U$, so by (3), $v(f(x)) \leq v(d)$. Similarly, by (3), $v(c) = v(f(b)) \leq v(d)$. Hence, $v\big(\frac{c}{f(x)}\big) \geq v(c) - v(d)$. Therefore,

$$\begin{aligned} v\Big(e\big(1 - \frac{c}{f(x)}\big)\Big) &\geq v(e) + \min(v(1), v(c) - v(d)) \\ &> \big(2v(d) - v(c)\big) + \big(v(c) - v(d)\big) = v(d). \end{aligned}$$

Thus,

$$e\Big(1 - \frac{c}{f(K)}\Big) \subseteq U. \tag{4}$$

Set $h(X,Y) = f(Y)\big(1 - \frac{f(X)}{e}\big) - c$. Since $f(Y)$ has no multiple roots and $c \neq 0$, Eisenstein's criterion [FrJ05, Lemma 2.3.10(b')] implies that $h(X,Y)$ is absolutely irreducible. By assumption, for each $K' \in \mathcal{X}$ there exists $a \in K'$ with $f(a) = 0$. Hence, $h(a,b) = 0$ and $\frac{\partial h}{\partial Y}(a,b) = f'(b) \neq 0$. Since K is P$\mathcal{X}$C, there are $x, y \in K$ with $h(x,y) = 0$. Thus, $f(x) = e\big(1 - \frac{c}{f(y)}\big)$. By (4), the right hand side is in U. Hence, $f(x) \in f(K) \cap U$. This contradiction to (2) completes the proof of (a).

Proof of (b): Assume no $K' \in \mathcal{X}$ is K-embeddable in $K_{v,\mathrm{alg}}$. Consider $K' \in \mathcal{X}$. By Lemma 2.1, there is an $a_{K'} \in K'$ such that $\mathrm{irr}(a_{K'}, K)$ has no roots in $K_{v,\mathrm{alg}}$. By definition, $\mathrm{SepAlgExt}(K(a_{K'}))$ is an étale open neighborhood of K' in $\mathrm{SepAlgExt}(K)$. The union of all these neighborhoods covers $\mathcal{X}$. Since $\mathcal{X}$ is étale compact, there are $K'_1, \ldots, K'_n \in \mathcal{X}$ with $\mathcal{X} \subseteq \bigcup_{i=1}^n \mathrm{SepAlgExt}(K(a_{K'_i}))$. Put $f(X) = \mathrm{lcm}(\mathrm{irr}(a_{K'_i}, K) \mid i = 1, \ldots, n)$. It is a separable polynomial without roots in $K_{v,\mathrm{alg}}$.

On the other hand, for each $K' \in \mathcal{X}$ there is an i with $K' \in \mathrm{SepAlgExt}(K(a_{K'_i}))$. Thus, $a_{K'_i}$ is a root of $f(X)$ in K'. We conclude from (a) that $f(X)$ has a root in $K_{v,\mathrm{alg}}$. This contradiction to the preceding paragraph proves there is a $K' \in \mathcal{X}$ which is K-embeddable in $K_{v,\mathrm{alg}}$. ∎

Call a pair (F, T) a **locality** if F is a field, and either T is the topology defined on F by a Henselian valuation or F is real closed and T is the topology defined by the unique ordering of F.

PROPOSITION 2.3 (Density [Pop90, Thm. 2.6]): *Let K be a field, $\mathcal{X}$ a $\mathrm{Gal}(K)$-invariant family of separable algebraic extensions of K, and (K', T') a locality. Suppose $\mathcal{X}$ is étale compact, K is $P\mathcal{X}C$, and K' is a minimal element of $\mathcal{X}$. Then:*

(a) *K is T'-dense in K'. Moreover, if T' is defined by a Henselian valuation v' of K' and $v = v'|_K$, then (K', v') is a Henselian closure of (K, v) and $K' = K_{v,\mathrm{alg}}$.*

(b) *Suppose $K' \neq K_s$. Then, $\mathrm{Aut}(K'/K) = 1$.*

(c) *Let (K'', T'') be a locality such that K'' is a minimal element of $\mathcal{X}$ and $K'' \neq K'$. Then $K'K'' = K_s$.*

Proof of (a): First we suppose T' is defined by a Henselian valuation v' of K'. Since (K', v') is Henselian, it contains a Henselian closure (K_v, v_h) of (K, v).

Lemma 2.2(b) gives $E \in \mathcal{X}$ in $K_{v,\mathrm{alg}}$. Thus, $K \subseteq E \subseteq K_{v,\mathrm{alg}} \subseteq K_v \subseteq K'$. Since K' is minimal, $E = K'$. Hence, $K_{v,\mathrm{alg}} = K_v = K'$. In particular, (K', v') is a Henselian closure of (K, v) and K is v'-dense in K' (Remark 1.6).

Now we suppose K' is real closed and T' is the topology defined by the unique ordering $<$ of K'. If $<$ is archimedean, then K' is contained in $\mathbb{R}$ and $\mathbb{Q} \subseteq K$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, so is K.

Suppose $<$ is nonarchimedean. Then the set of all $x \in K'$ with $-n \leq x \leq n$ for some $n \in \mathbb{N}$ is a valuation ring of a Henselian valuation v of K' [Jar91b, Lemma 16.2]. In particular, $\{x \in K' \mid -1 \leq x \leq 1\} \subseteq O_v$. This means, in the terminology of [Jar91b, §16], v is **coarser** than $<$. It follows from [Jar91b, Remark 16.3] that the T'-topology on K' coincides with the $<$-topology. We conclude from the first case that K is $<$-dense in K'.

Proof of (b): Statement (b) is well known if K' is real closed [Lan, p. 455, Thm. 2.9]. When T' is defined by a Henselian valuation v of K', use (a) and Proposition 1.14.

Proof of (c): If K' or K'' is real closed, then its codegree in $\tilde{K}$ is 2. Hence, $K' \neq K''$ implies $K'K'' = \tilde{K}$. If both T' and T'' are induced by nontrivial valuations, use (a) and Proposition 1.11. ∎

The minimality condition in Proposition 2.3 is automatic if (K', v') is a Henselian closure of (K, v) with a finite residue field.

LEMMA 2.4: *Let E be a Henselian field with respect to valuations v and w. Suppose both $\bar{E}_v$ and $\bar{E}_w$ are algebraic extensions of finite fields but at least one of them is not algebraically closed. Then v and w are equivalent.*

Proof: Since one of the fields $\bar{E}_v$ and $\bar{E}_w$ is not separably closed, v and w are comparable [Jar91b, Prop. 13.4]. This means $\bar{E}_v$ is a residue field of $\bar{E}_w$ or $\bar{E}_w$ is a residue field of $\bar{E}_v$. Since both $\bar{E}_v$ and $\bar{E}_w$ are algebraic extensions of finite fields, none of them has a nontrivial valuation. Hence, $\bar{E}_v = \bar{E}_w$. Consequently, v and w are equivalent. ∎

PROPOSITION 2.5: *Let K be a field and X a set. For each $x \in X$ let (K_x, v_x) a valued field with residue field $\bar{K}_x$. Suppose $\bar{K}_x$ is finite and (K_x, v_x) is the Henselian closure of $(K, v_x|_K)$ for all $x \in X$, and $\mathcal{X} = \{K_x^\sigma \mid x \in X,\ \sigma \in \mathrm{Gal}(K)\}$ is étale compact, and K is $P\mathcal{X}C$. Then K is v_x-dense in K_x.*

Proof: By Proposition 2.3, it suffices to prove each K_x with $x \in X$ is minimal in $\mathcal{X}$.

Let $y \in X$, $\sigma \in \mathrm{Gal}(K)$, and $K_y^\sigma \subseteq K_x$. We may assume that $\sigma = 1$. Extend v_y to a valuation v_y' of K_x. Then K_x is Henselian with respect to both v_x and v_y'. By assumption, $\bar{K}_x$ is finite and $(\overline{K_x})_{v_y'}$ is an algebraic extension of the finite field $\bar{K}_y$. By Lemma 2.4, $v_x = v_y'$, so $v_x|_K = v_y|_K$. Thus, both K_x and K_y are Henselian closures of K with respect to the same valuation, hence $K_x \cong_K K_y$. Since $K \subseteq K_y \subseteq K_x$, this implies $K_x = K_y$ [FrJ05, Lemma 20.6.2]. ∎

3. The Bounded Operator

Let (K, v) be a Henselian field having an element π with a minimal positive value and a finite residue field of q elements. The **Kochen operator**

$$\gamma(X) = \frac{1}{\pi} \frac{X^q - X}{(X^q - X)^2 - 1}$$

is then a rational function on K satisfying $\gamma(K) = O_v$ [JaR80, p. 426]. It plays a central role in the theory of P-adically closed fields.

Here we consider an arbitrary valued field (K, v) with a finite residue field. Let m be a positive integer. Denote the residue of an element $a \in O_v$ (resp. polynomial $g \in O_v[X]$) in $\bar{K}_v$ (resp. $\bar{K}_v[X]$) by $\bar{a}$ (resp. $\bar{g}$). We say (K, v) has an m**-bounded** residue field if $\bar{K}_v$ is finite and m is a multiple of $|\bar{K}_v^\times|$. Then $\bar{a}^m = 1$ for each $a \in O_v$ with $v(a) = 0$.

We replace the Kochen operator by the m**-bounded operator**

$$\wp(X) = \wp_m(X) = \frac{X^{2m}}{X^{2m} - X^m + 1} \in K(X). \tag{1}$$

It has several improved properties which turn out to be useful in the proof of the block approximation theorem:

LEMMA 3.1: *Let (K, v) be a valued field with an m-bounded residue field. Then the following holds for each $a \in K$:*

(a) $v(\wp(a)) \geq 0$.

(b) $v(\wp(a)) \geq v(a)$.

(c) *Either* $v(\wp(a)) > 0$ *or* $v(\wp(a) - 1) > 0$.

(d) $v(a) > 0$ *if and only if* $v(\wp(a)) > 0$.

(e) $v(a) \leq 0$ *if and only if* $v(\wp(a) - 1) > 0$.

Proof: The assertions follow by analyzing the three possible cases.

First we suppose, $v(a) > 0$. Then $v(a^{2m} - a^m + 1) = v(1) = 0$, so $v(\wp(a)) = 2mv(a) > v(a) > 0$.

Now suppose $v(a) < 0$. Then $v(a^{2m} - a^m + 1) = v(a^{2m}) = 2mv(a)$ and $v(a^m - 1) = mv(a)$. Thus $v(\wp(a) - 1) = v(a^m - 1) - v(a^{2m} - a^m + 1) = -mv(a) > 0$ and $v(\wp(a)) = 0$.

Finally suppose $v(a) = 0$. Then $v(a^{2m} - a^m + 1) \geq 0$. Hence, $\bar{a}^{2m} - \bar{a}^m + 1 = 1^2 - 1 + 1 \neq 0$. Therefore, $v(a^{2m} - a^m + 1) = 0$. Consequently, $v(\wp(a) - 1) = v(a^m - 1) > 0$ and $v(\wp(a)) = 0$. ∎

Lemma 3.1(a),(b) implies:

COROLLARY 3.2: *Given $c_1, \ldots, c_r \in K^\times$, the element*

$$c = \prod_{i=1}^{r} \wp(c_i)$$

satisfies $v(c) \geq v(c_1), \ldots, v(c_r)$.

Notation 3.3: A special rational function. Consider a polynomial

$$g(Y) = b_n Y^n + b_{n-1} Y^{n-1} + \cdots + b_1 Y + b_0 \in O_v[Y]$$

satisfying the following conditions:

(3a) $b_0, b_n \in \mathcal{O}_v^\times$,

(3b) $\bar{g}(Y)$ has no roots in $\bar{K}_v = \mathbb{F}_q$,

(3c) if $\mathrm{char}(K) = p > 0$, then g has a root of multiplicity $< p$ in $\tilde{K}$, and

(3d) $n \geq 4$.

(For instance, $g(Y) = \frac{Y^m - 1}{Y - 1} = Y^{m-1} + \cdots + Y + 1$, where $m \geq 5$ is relatively prime to $q(q-1)$. In this case each zero of g in $\tilde{K}$ is simple.) Set

$$f(Y) = b_1 Y^3 - 2b_1 Y^2 + (b_1 - b_0) Y + b_0$$

and

$$\begin{aligned} \Phi(Y) &= 1 - \frac{f(Y)}{g(Y)} = \frac{g(Y) - f(Y)}{g(Y)} \\ &= \frac{b_n Y^n + \cdots + b_4 Y^4 + (b_3 - b_1) Y^3 + (b_2 + 2b_1) Y^2 + b_0 Y}{b_n Y^n + b_{n-1} Y^{n-1} + \cdots + b_1 Y + b_0}. \quad ∎ \end{aligned}$$

LEMMA 3.4: *Let (K, v) be a valued field with an m-bounded residue field.*

(a) *Every $a \in K$ satisfies $v(\Phi(a)) \geq 0$. Moreover, if $v(a) > 0$ then $v(\Phi(a)) \geq v(a)$.*

(b) *$\Phi(0) = 0$ and $\Phi(1) = 1$.*

(c) *Suppose (K, v) is Henselian. Let $c \in K$ be such that either $v(c) > 0$ or $v(c-1) > 0$. Then there is a $y \in K$ such that $\Phi(y) = c$ and $\Phi'(y) \neq 0$.*

(d) *The numerator of the rational function $\Phi(Y_1) + \Phi(Y_2) + \Phi(Y_3) - a$ is absolutely irreducible for each $a \in K$.*

Proof of (a): If $v(a) > 0$, then $v(g(a)) = v(b_0) = 0$. Also, $v(g(a) - f(a)) \geq v(a)$, because $Y | \big(g(Y) - f(Y)\big)$. Hence $v(\Phi(a)) \geq v(a) > 0$.

If $v(a) = 0$, then $v(g(a) - f(a)) \geq 0$ and $v(g(a)) \geq 0$. But $v(g(a)) \not> 0$, by (3b). Hence, $v(g(a)) = 0$. Therefore, $v(\Phi(a)) \geq 0$.

Finally, if $v(a) < 0$, then $v(g(a) - f(a)) = nv(a)$ and $v(g(a)) = nv(a)$. Hence $v(\Phi(a)) = 0$.

Proof of (c): It suffices to show that the polynomial

$$h(Y) = f(Y) + (c-1)g(Y) \in K[Y]$$

has a root y in K such that $h'(y) \neq 0$ and $g(y) \neq 0$. Indeed, then $\Phi(y) = c$ and $\Phi'(y) = -\frac{h'(y)}{g(y)}$. By (3b) it suffices to find a root $y \in \mathcal{O}_v$ of h such that $h'(y) \neq 0$.

If $v(c) > 0$, then

(4a) $h(0) \equiv f(0) - g(0) \equiv b_0 - b_0 \equiv 0 \mod \mathfrak{m}_v$ and

(4b) $h'(0) \equiv f'(0) - g'(0) \equiv (b_1 - b_0) - b_1 \equiv -b_0 \not\equiv 0 \mod \mathfrak{m}_v$.

If $v(c-1) > 0$, then

(5a) $h(1) \equiv f(1) \equiv b_1 - 2b_1 + (b_1 - b_0) + b_0 \equiv 0 \mod \mathfrak{m}_v$ and

(5b) $h'(1) \equiv f'(1) \equiv 3b_1 - 4b_1 + (b_1 - b_0) \equiv -b_0 \not\equiv 0 \mod \mathfrak{m}_v$.

Thus, the assertion follows from Hensel's Lemma.

Proof of (d): By [Gey94, Theorem A], it suffices to prove the following statement: Suppose $\mathrm{char}(K) = p > 0$. Then there exist no rational function $\Psi(Y) \in \tilde{K}(Y)$ and $a_0, a_1, \ldots, a_k \in \tilde{K}$ with $k > 0$, $a_k \neq 0$, and $\Phi(Y) = \sum_{j=0}^{k} a_j \Psi^{p^j}(Y)$.

Assume the contrary. Then every pole of $\Phi(Y)$ is a pole of $\Psi(Y)$. Conversely, every pole of $\Psi(Y)$, say, of order d, is a pole of $\Phi(Y)$ of order $p^k d$. Thus, every pole of $\Phi(Y)$ is of order divisible by p. Hence, every zero of $g(Y)$ is of order $\geq p$. This contradicts (3c). ∎

Lemma 3.5: *Let $V \subseteq \mathbb{A}^n$ be an absolutely irreducible affine variety, defined over a field K by polynomials $f_1, \ldots, f_m \in K[\mathbf{X}] = K[X_1, \ldots, X_n]$. Set $K[\mathbf{x}] = K[\mathbf{X}]/(f_1, \ldots, f_m)$. Let $r \geq 0$ and for each $1 \leq i \leq r$ let $h_i \in K[\mathbf{X}, \mathbf{Y}_i] = K[X_1, \ldots, X_n, Y_{i1}, \ldots, Y_{in_i}]$ be a polynomial such that $h_i(\mathbf{x}, \mathbf{Y}_i) \in K[\mathbf{x}, \mathbf{Y}_i]$ is absolutely irreducible. Suppose the tuples $\mathbf{X}, \mathbf{Y}_1, \ldots, \mathbf{Y}_r$ are disjoint. Then the affine variety W defined in $\mathbb{A}^{m+n_1+\cdots+n_r}$ by the equations*

$$f_i(\mathbf{X}) = 0,\ i = 1, \ldots, m; \quad h_j(\mathbf{X}, \mathbf{Y}_j) = 0,\ j = 1, \ldots, r,$$

is an absolutely irreducible variety defined over K of dimension $\dim(V) + (n_1 - 1) + \cdots + (n_r - 1)$.

Proof: For each $1 \leq i \leq r$ put

$$R_i = \tilde{K}[\mathbf{X}, \mathbf{Y}_1, \ldots, \mathbf{Y}_i]/(f_1(\mathbf{X}), \ldots, f_m(\mathbf{X}), h_1(\mathbf{X}, \mathbf{Y}_1), \ldots, h_i(\mathbf{X}, \mathbf{Y}_i));$$

if R_i is a domain, let Q_i be its quotient field. Put $d_i = \dim(V) + (n_1 - 1) + \cdots + (n_i - 1)$.

We have to show that $\tilde{K}[W] = R_r$ is an integral domain and $\mathrm{trans.deg}_{\tilde{K}} Q_r = d_r$.

Observe that $R_0 = \tilde{K}[V]$ and $\mathrm{trans.deg}_{\tilde{K}} Q_0 = d_0$. Suppose, by induction on i, that R_{i-1} is a domain and $\mathrm{trans.deg}_{\tilde{K}} Q_{i-1} = d_{i-1}$. Since $h_i(\mathbf{x}, \mathbf{Y}_i)$ is irreducible over Q_{i-1}, the ring $Q_{i-1}[\mathbf{Y}_i]/(h_i(\mathbf{x}, \mathbf{Y}_i))$ is a domain. Hence so is its subring $R_i = R_{i-1}[\mathbf{Y}_i]/(h_i(\mathbf{x}, \mathbf{Y}_i))$ and Q_i is the quotient field of $Q_{i-1}[\mathbf{Y}_i]/(h_i(\mathbf{x}, \mathbf{Y}_i)$. We conclude that $\mathrm{trans.deg}_{\tilde{K}} Q_i = d_{i-1} + (n_i - 1) = d_i$. ∎

Definition 3.6: Let K be a field. The **patch topology** of $\mathrm{Val}(K)$ has a basis consisting of all open-closed sets of the form

$$\{v \in \mathrm{Val}(K) \mid\ v(b_1) > 0, \ldots, v(b_{k_0}) > 0,\ v(b_{k_0+1}) \geq 0, \ldots, v(b_k) \geq 0\}, \tag{6}$$

with $v_1, \ldots, b_k \in K$ [HJP07, Section 8]. In particular, each of the sets

$$\{v \in \mathrm{Val}(K) \mid\ v(c_1) = 0, \ldots, v(c_m) = 0\}$$

with $c_1, \ldots, c_m \in K^\times$ is open-closed in $\mathrm{Val}(K)$. By [HJP07, Prop. 8.2], $\mathrm{Val}(K)$ is profinite under the patch topology. Let $\mathbf{B}$ be a closed subset of $\mathrm{Val}(K)$, and m a positive integer. We say $\mathbf{B}$ has **m-bounded residue fields** if each (K, v) with $v \in \mathbf{B}$ has an m-bounded residue field. ∎

Lemma 3.7: *Let K be a field and $\mathbf{B}$ a closed subset of* $\mathrm{Val}(K)$ *with m-bounded residue fields. Let $\mathbf{B}_0$ be an open-closed subset of $\mathbf{B}$. Then there exists $b \in K$ such that $\mathbf{B}_0 = \{v \in \mathbf{B} \mid v(b) > 0\}$ and $\mathbf{B} \smallsetminus \mathbf{B}_0 = \{v \in \mathbf{B} \mid v(1-b) > 0\}$.*

Proof: First we assume that $\mathbf{B}_0$ is the intersection of $\mathbf{B}$ with a basic set of the form (6). By Lemma 3.1(e), for each $k_0 + 1 \le j \le k$ the condition $v(b_j) \ge 0$ is equivalent to $v(1 - \wp(b_j^{-1})) > 0$. Thus, we may assume that

$$\mathbf{B}_0 = \{v \in \mathbf{B} \mid v(b_1) > 0, \ldots, v(b_k) > 0\}.$$

If $k = 0$, then $\mathbf{B}_0 = \mathbf{B}$ and we may take $b = 0$. Thus, we assume that $k \ge 1$.

By Lemma 3.1(d), we may replace b_j by $\wp(b_j)$. Hence, by Lemma 3.1(c), we may assume that, for each $v \in \mathbf{B}$, either $v(b_j) > 0$ or $v(1 - b_j) > 0$. For $k = 1$, this gives $\mathbf{B}_0 = \{v \in \mathbf{B} \mid v(b_1) > 0\}$ and $\mathbf{B} \smallsetminus \mathbf{B}_0 = \{v \in \mathbf{B} \mid v(1 - b_1) > 0\}$.

Suppose $k = 2$. Then $\mathbf{B}_0 = \{v \in \mathbf{B} \mid v(b_1) > 0,\ v(b_2) > 0\}$ and each $v \in \mathbf{B} \smallsetminus \mathbf{B}_0$ satisfies either $b_1 \equiv 1 \mod \mathfrak{m}_v$ and $b_2 \equiv 0 \mod \mathfrak{m}_v$, or $b_1 \equiv 1 \mod \mathfrak{m}_v$ and $b_2 \equiv 1 \mod \mathfrak{m}_v$, or $b_1 \equiv 0 \mod \mathfrak{m}_v$ and $b_2 \equiv 1 \mod \mathfrak{m}_v$. Set $b = b_1^2 - b_1 b_2 + b_2^2$. Then $v(b) > 0$ for each $v \in \mathbf{B}_0$ and $v(1 - b) > 0$ for each $v \in \mathbf{B} \smallsetminus \mathbf{B}_0$. The quickest way to check the latter relation is to prove that $b \equiv 1 \mod \mathfrak{m}_v$ by computing b modulo $\mathfrak{m}_v$ in each of the above mentioned three alternatives.

If $k \ge 3$, we inductively find b_1' such that $\mathbf{B}_0 = \{v \in \mathbf{B} \mid v(b_1') > 0,\ v(b_k) > 0\}$. Then we use the case $k = 2$.

In the general case $\mathbf{B}_0$ is compact, and hence it is a finite union of basic subsets of $\mathbf{B}$. The preceding paragraphs prove that the collection of subsets of the required form contains the basic subsets and is closed under finite intersections. Clearly it is also closed under taking complements in $\mathbf{B}$: if $\mathbf{B}_0$ is defined by b then $\mathbf{B} \smallsetminus \mathbf{B}_0$ is defined by $1 - b$. Therefore this collection is closed also under finite unions. ∎

Lemma 3.8: *Let K be an infinite field and $\mathbf{B}$ a closed subset of* $\mathrm{Val}(K)$ *with m-bounded residue fields. Then there is a $b \in K^\times$ such that $v(b) > 0$ for all $v \in \mathbf{B}$.*

Proof: First we note that the residue field of the trivial valuation v_0 of K is K itself, hence v_0 is not m-bounded, so $v_0 \notin \mathbf{B}$. Therefore, for each $v \in \mathbf{B}$ there is a $b_v \in K^\times$

such that $v(b_v) > 0$. If $v' \in \mathbf{B}$ is sufficiently close to v, then also $v'(b_v) > 0$. Since $\mathbf{B}$ is compact, there are $b_1, \ldots, b_r \in K^\times$ such that for each $v \in \mathbf{B}$ there is an $i = i(v)$ with $v(b_i) > 0$. By Corollary 3.2, $b = \prod_i \wp(b_i)$ satisfies $v(b) \geq v(b_{i(v)}) > 0$ for each $v \in \mathbf{B}$.

∎

4. Block Approximation Theorem

The block approximation theorem is a far reaching generalization of the weak approximation theorem. The latter deals with independent valuations $v_1, \ldots, v_n$ of a field K and elements $a_1, \ldots, a_n \in K$ and $c_1, \ldots, c_n \in K^\times$. It assures the existence of $a \in K$ with $v_i(a - a_i) > v(c_i)$, $i = 1, \ldots, n$. The block approximation theorem considers a family (K_x, v_x) of valued fields with K_x separable algebraic over K indexed by a profinite space X and an affine variety V. The space X is partitioned into finitely many "blocks" X_i. For each i a point $\mathbf{a}_i \in \bigcap_{x \in X_i} V_{\mathrm{simp}}(K_x)$ and an element $c_i \in K^\times$ are given. Under certain conditions on this data, the block approximation theorem gives an $\mathbf{a} \in V(K)$ such that $v_x(\mathbf{a} - \mathbf{a}_i) > v_x(c_i)$ for all i and each $x \in X_i$.

The version of the block approximation theorem we prove assumes that the residue fields of (K_x, v_x) are finite with bounded cardinality. The formulation of all other conditions uses terminology of [HJP07] which we now recall.

Let G be a profinite group. Denote the set of all closed subgroups of G by $\mathrm{Subgr}(G)$. This set is equipped with two topologies, the **strict topology** and the **étale topology**. A basic **strict open neighborhood** of an element H_0 of $\mathrm{Subgr}(G)$ is the set $\{H \in \mathrm{Subgr}(G) \mid HN = H_0N\}$, where N is an open normal subgroup of G. A basic **étale open neighborhood** of $\mathrm{Subgr}(G)$ is the set $\mathrm{Subgr}(G_0)$, where G_0 is an open subgroup of G. See also [HJP07, §1] and [HJP05, Section 2] for more details.

Now let K be a field. Denote the set of all algebraic (resp. separable algebraic) extensions of K by $\mathrm{AlgExt}(K)$ (resp. $\mathrm{SepAlgExt}(K)$). Galois correspondence carries over the strict and the étale topologies of $\mathrm{Gal}(K)$ to strict and étale topologies of $\mathrm{SepAlgExt}(K)$. Thus, a **basic strict open neighborhood** of an element F_0 of $\mathrm{SepAlgExt}(K)$ is the set $\{F \in \mathrm{SepAlgExt}(K) \mid F \cap N = F_0 \cap N\}$, where N is a finite Galois extension of K. A **basic étale open neighborhood** of $\mathrm{SepAlgExt}(K)$ is the set $\mathrm{SepAlgExt}(L)$, where L is a finite separable extension of K.

A **group-structure** is a system $\mathbf{G} = (G, X, G_x)_{x \in X}$ consisting of a profinite group acting continuously (from the right) on a profinite space X and a closed subgroup G_x of G for each $x \in X$ satisfying these conditions:

(1a) The map $\delta\colon X \to \mathrm{Subgr}(G)$ defined by $\delta(x) = G_x$ is étale continuous.

(1b) $G_{x^\sigma} = G_x^\sigma$ for all $x \in X$ and $\sigma \in G$

(1c) $\{\sigma \in G \mid x^\sigma = x\} \leq G_x$ [HJP07, §2].

A **special partition** of a group-structure $\mathbf{G}$ as above is a data $(G_i, X_i)_{i \in I_0}$ satisfying the following conditions [HJP07, Def. 3.5]:

(2a) I_0 is a finite set disjoint from X.

(2b) X_i is a nonempty open-closed subset of X, $i \in I_0$.

(2c) G_i is an open subgroup of G containing G_x for all $x \in X_i$ and $i \in I_0$.

(2d) $G_i = \{\sigma \in G \mid X_i^\sigma = X_i\}$, $i \in I_0$.

(2e) For each $i \in I_0$ let R_i be a subset of G satisfying $G = \dot{\bigcup}_{\rho \in R_i} G_i \rho$. Then $X = \dot{\bigcup}_{i \in I_0} \dot{\bigcup}_{\rho \in R_i} X_i^\rho$.

A **proper field-valuation-structure** is a system $\mathbf{K} = (K, X, K_x, v_x)_{x \in X}$, where K is a field, X is a profinite space, and for each $x \in X$ K_x is a separable algebraic extension of K and v_x is a valuation of K_x satisfying these conditions:

(3a) Let $\mathcal{X} = \{K_x \in \mathrm{SepAlgExt}(K) \mid x \in X\}$ and $\delta\colon X \to \mathcal{X}$ the maps defined by $\delta(x) = K_x$. Then δ is an étale homeomorphism. In particular, $\mathcal{X}$ is étale profinite.

(3b) $K_x^\sigma = K_{x^\sigma}$ and $v_x^\sigma = v_{x^\sigma}$ for all $x \in X$ and $\sigma \in \mathrm{Gal}(K)$.

(3c) $x^\sigma = x$ implies $\sigma \in \mathrm{Gal}(K_x)$ for all $x \in X$ and $\sigma \in \mathrm{Gal}(K)$.

(3d) For each finite separable extension L of K let $X_L = \{x \in X \mid L \subseteq K_x\}$. Then the map $\nu_L\colon X_L \to \mathrm{Val}(L)$ defined by $\nu_L(x) = v_x|_L$ is continuous.

In particular, $\mathrm{Gal}(\mathbf{K}) = (\mathrm{Gal}(K), X, \mathrm{Gal}(K_x))_{x \in X}$ is a group structure.

A **block approximation problem** for a proper field-valuation-structure $\mathbf{K}$ is a data $(V, X_i, L_i, \mathbf{a}_i, c_i)_{i \in I_0}$ satisfying the following conditions:

(4a) $(\mathrm{Gal}(L_i), X_i)_{i \in I_0}$ is a special partition of $\mathrm{Gal}(\mathbf{K})$.

(4b) V is an affine absolutely irreducible variety over K.

(4c) $\mathbf{a}_i \in V_{\mathrm{simp}}(L_i)$.

(4d) $c_i \in K^\times$.

A **solution** of the problem is a point $\mathbf{a} \in V(K)$ satisfying $v_x(\mathbf{a} - \mathbf{a}_i) > v_x(c_i)$ for all $i \in I_0$ and $x \in X_i$. We say $\mathbf{K}$ satisfies the **block approximation condition** if each block approximation problem has a solution.

Theorem 4.1 (Residue Bounded Block Approximation Theorem):
Let $\mathbf{K} = (K, X, K_x, v_x)_{x\in X}$ *be a proper field-valuation-structure. Put* $\mathcal{X} = \{K_x \mid x \in X\}$ *and* $\mathbf{B} = \{v_x|_K \mid x \in X\}$. *Suppose* K *is* $P\mathcal{X}C$, $\mathbf{B}$ *is* m*-bounded for some positive integer* m, *and for all* $x \in X$ *the valued field* (K_x, v_x) *is the Henselian closure of* $(K, v_x|_K)$. *Then* $\mathbf{K}$ *has the block approximation property.*

Proof: We let (4) be a block approximation problem for $\mathbf{K}$ and divide the rest of the proof into several parts.

Part A: *Proof in case* $V = \mathbb{A}^1$. We write a, a_i rather than $\mathbf{a}, \mathbf{a}_i$, respectively.

Part A1: *Reduction to the case where* $a_i \in K$, *for all* $i \in I_0$. Fix $i \in I_0$. Let $x \in X_i$. By Proposition 2.5, K is v_x-dense in K_x. Hence, there is $a_{ix} \in K$ with $v_x(a_{ix} - a_i) > v_x(c_i)$. We consider the open-closed subset $T_{ix} = \{w \in \mathrm{Val}(L_i) \mid w(a_{ix} - a_i) > w(c_i)\}$ of $\mathrm{Val}(L_i)$. By (2c), $L_i \subseteq K_x$ for each $x \in X_i$. By (3d), the map $X_i \to \mathrm{Val}(L_i)$ defined by $y \mapsto v_y|_{L_i}$ is continuous. Hence, $X_{ix} = \{y \in X_i \mid v_y(a_{ix} - a_i) > v_y(c_i)\}$, which is the inverse image of T_{ix} in X_i, is an open-closed neighborhood of x in X_i. Moreover, X_{ix} is $\mathrm{Gal}(L_i)$-invariant.

Since X_i is compact, finitely many of these neighborhoods cover X_i. Hence, there is a partition $X_i = X_{i1} \cup \cdots \cup X_{it}$ of X_i with X_{ij} closed and $\mathrm{Gal}(L_i)$-invariant and for each $1 \le j \le t$ there is some $a_{ij} \in K$ with $v_x(a_{ij} - a_i) > v_x(c_i)$ for all $x \in X_{ij}$.

If we find $a \in K$ with $v_x(a - a_{ij}) > v_x(c_i)$ for all $x \in X_{ij}$, with $i \in I_0$, then $v_x(a - a_i) > v_x(c_i)$ for all $x \in X_{ij}$ with $i \in I_0$. Thus, replacing the family $\{X_i \mid i \in I_0\}$ by its refinement $\{X_{ij} \mid i, j\}$, and the elements a_i by a_{ij}, if necessary, we may assume $a_i \in K$.

Part A2: *Reduction to the case where* $L_i = K$. Let $\mathbf{B}_i = \{v_x|_K \mid x \in X_i\}$. Since the map $X \to \mathrm{Val}(K)$ is continuous (by (3d)) and both X and $\mathrm{Val}(K)$ are profinite spaces (Definition 3.6), $\mathbf{B}$ and each of the sets $\mathbf{B}_i$ is closed in $\mathrm{Val}(K)$. If $x \in X_i$ and $\rho \in \mathrm{Gal}(K)$, then $v_{x^\rho}|_K = v_x^\rho|_K = v_x$, hence $\mathbf{B} = \bigcup_{i\in I_0} \mathbf{B}_i$ (by (2e)). Moreover, $\mathbf{B} = \dot{\bigcup}_{i\in I_0} \mathbf{B}_i$. Indeed, assume there are distinct $i, j \in I_0$ and $x \in X_i$, $x' \in X_j$ with $v_x|_K = v_{x'}|_K$. Then there exists $\sigma \in \mathrm{Gal}(K)$ with $(K_x^\sigma, v_x^\sigma) = (K_{x'}, v_{x'})$ (because both (K_x, v_x) and $(K_{x'}, v_{x'})$ are Henselian closure of $(K, v_x|_K)$). By (3b), $K_{x^\sigma} = K_{x'}$. Hence,

by (3a), $x^\sigma = x'$. Let $\rho \in R_i$ and $\tau \in \mathrm{Gal}(L_i)$ with $\sigma = \tau\rho$. Then $x' = x^{\tau\rho} \in X_i^{\tau\rho} = X_i^\rho$. This is a contradiction to $X = \dot{\bigcup}_{i\in I_0} \dot{\bigcup}_{\rho\in R_i} X_i^\rho$ (Assumption (2e)). It follows that each of the sets $\mathbf{B}_i$ is open-closed in $\mathbf{B}$.

Thus, we have to find an $a \in K$ with $v(a - a_i) > v(c_i)$ for all $i \in I_0$ and $v \in \mathbf{B}_i$.

Part A3: *Simplifying* $\mathbf{B}_i$. If there is an i with $\mathbf{B}_i = \mathbf{B}$ (and hence $\mathbf{B}_j = \emptyset$ for $j \neq i$), take $a = a_i$. Thus, we may assume $\mathbf{B}_i \neq \mathbf{B}$ for each i.

Since $\mathbf{B}$ is closed in $\mathrm{Val}(K)$ and each $\mathbf{B}_i$ is open-closed in $\mathbf{B}$ (Part A2), Lemma 3.7 gives for each i an element $d_i \in K$ with $\mathbf{B}_i = \{v \in \mathbf{B} \mid v(d_i) > 0\}$ and $\mathbf{B} \smallsetminus \mathbf{B}_i = \{v \in \mathbf{B} \mid v(1 - d_i) > 0\}$. Since $\mathbf{B}_i \neq \mathbf{B}$, we have $d_i \neq 0$.

Part A4: *System of equations.* Let $\wp = \wp_m$ be the m-bounded operator defined by (1) of Section 3. We write I_0 as $\{1, 2, \ldots, r\}$ and consider the system

$$\wp\Big(\frac{Z - a_i}{c_i}\Big) = d_i\big(\Phi(Y_{i1}) + \Phi(Y_{i2}) + \Phi(Y_{i3})\big), \qquad i = 1, \ldots, r \tag{5}$$

of r equations in $3r + 1$ variables Z, Y_{ij}, where Φ is the special rational function defined in Notation 3.3. By Lemma 3.4(d), each of these equations is absolutely irreducible over the field of rational functions $K(Z)$. Therefore, by Lemma 3.5, with $V = \mathbb{A}^1$, (5) defines an absolutely irreducible variety $A \subseteq \mathbb{A}^{3r+1}$ over K of dimension $2r + 1$.

Part A5: *Local solution.* Let $v \in \mathbf{B}$. There is a unique $k \in I_0$ with $v \in \mathbf{B}_k$. We choose an $a \in K_v$ with $v(a - a_k) > v(d_k) + v(c_k)$. Then $v(\frac{a-a_k}{c_k}) > v(d_k)$. Hence, by Lemma 3.1(b), $v(\wp(\frac{a-a_k}{c_k})) > v(d_k)$, so $v(d_k^{-1}\wp(\frac{a-a_k}{c_k})) > 0$. By Lemma 3.4(c), there is a $b_{k1} \in K_v$ such that $\Phi(b_{k1}) = d_k^{-1}\wp(\frac{a-a_k}{c_k})$ and $\Phi'(b_{k1}) \neq 0$. By Lemma 3.4(b), $\Phi(0) = 0$. Therefore, $(a, b_{k1}, 0, 0)$ solves the kth equation of (5).

Let $j \in I_0$ such that $j \neq k$. By Lemma 3.1(c), either $v(\wp(\frac{a-a_j}{c_j})) > 0$ or $v(\wp(\frac{a-a_j}{c_j}) - 1) > 0$. Since $v \notin \mathbf{B}_j$, we have $v(d_j - 1) > 0$ (Part A3), so $v(d_j) = 0$. Therefore, either $v\big(d_j^{-1}\wp(\frac{a-a_i}{c_i})\big) > 0$ or $v\big(d_j^{-1}\wp(\frac{a-a_i}{c_i}) - 1\big) > 0$. In both cases, Lemma 3.4(c) gives a $b_{j1} \in K_v$ with $\Phi(b_{j1}) = d_j^{-1}\wp\Big(\frac{a-a_j}{c_j}\Big)$ and $\Phi'(b_{j1}) \neq 0$. It follows that $(a, b_{j1}, 0, 0)$ is a solution of the jth equation of (5).

The solution $Z = a, Y_{i1} = b_{i1}, Y_{i2} = Y_{i3} = 0,\ i = 1, \ldots, r$, of (5) is a K_v-rational point on A. It is simple, because the corresponding $r \times (3r + 1)$ Jacobi matrix of

derivatives of the equations in (5) contains a submatrix of rank r. Indeed, the matrix of derivatives with respect to $Y_{11}, \ldots, Y_{r1}$, is the non-singular diagonal matrix

$$\mathrm{diag}\big(d_1\Phi'(b_{11}), \ldots, d_r\Phi'(b_{r1})\big).$$

Thus, (5) has a simple solution in K_v for each $v \in \mathbf{B}$.

Part A6: *Global solution.* Since K is P$\mathcal{X}$C, (5) has a K-rational solution $(a, \mathbf{b})$. Thus, $\wp\big(\frac{a-a_i}{c}\big) = d_i\big(\Phi(b_{i1}) + \Phi(b_{i2}) + \Phi(b_{i3})\big)$, $i = 1, \ldots, r$.

Let $1 \le i \in I_0$ and $v \in \mathbf{B}_i$. Then $v(d_i) > 0$ (Part A3). Hence, by Lemma 3.4(a), $v(\wp(\frac{a-a_i}{c_i})) \ge v(d_i) > 0$. By Lemma 3.1(d), $v(\frac{a-a_i}{c_i}) > 0$. Consequently, $v(a - a_i) > v(c_i)$.

Part B: *Proof of the general case.* If K is finite, then $\mathrm{Val}(K)$ consists of the trivial valuation only. The Henselization of K at that valuation is K itself. Hence, this is a trivial case, so we assume K is infinite.

Part B1: *System of equations.* Lemma 3.8 gives $b \in K^\times$ with $v(b) > 0$ for all $v \in \mathbf{B}$. Put $c = b\prod_{i\in I_0} \wp(c_i)$. By Lemma 3.1, $v(c) = v(b) + \sum_{j\in I_0} v(\wp(c_j)) > v(\wp(c_j)) > v(c_i)$ for each $i \in I_0$. By Part A, there is an $\mathbf{a}' = (a'_1, \ldots, a'_n) \in \mathbb{A}^n(K)$ with $v_x(a'_\nu - a_{i\nu}) > v_x(c_i)$ for each $i \in I_0$, each $1 \le \nu \le n$, and every $x \in X_i$. Thus, it suffices to find $\mathbf{a} \in V(K)$ with $v_x(a_\nu - a'_\nu) \ge v_x(c_i)$ for each $1 \le \nu \le n$ and for all $x \in X$.

Suppose V is defined by polynomials $f_1(\mathbf{Z}), \ldots, f_m(\mathbf{Z}) \in K[Z_1, \ldots, Z_n]$. Consider the Zariski-closed set $W \subseteq \mathbb{A}^{4n}$ defined over K by the equations

$$\begin{aligned} &f_\mu(\mathbf{Z}) = 0, \qquad \mu = 1, \ldots, m, \\ &\frac{Z_\nu - a'_\nu}{c} = \Phi(Y_{\nu 1}) + \Phi(Y_{\nu 2}) + \Phi(Y_{\nu 3}), \qquad \nu = 1, \ldots, n. \end{aligned} \tag{6}$$

Since V is absolutely irreducible, $K[\mathbf{z}] = K[\mathbf{Z}]/(f_1, \ldots, f_m)$ is an integral domain. By Lemma 3.4(d), with $\frac{z_\nu - a'_{i\nu}}{c}$ replacing a, each of the equations $\frac{z_\nu - a'_\nu}{c} = \Phi(Y_{\nu 1}) + \Phi(Y_{\nu 2}) + \Phi(Y_{\nu 3})$ is absolutely irreducible. Hence, by Lemma 3.5, W is an absolutely irreducible variety over K of dimension $\dim(V) + 2n$.

Part B2: *Rational points on W.* Let $x \in X_i$ for some $i \in I_0$ and $1 \le \nu \le n$. By Part B1, $v_x(\frac{a_{i\nu} - a'_\nu}{c}) > 0$. Hence, by Lemma 3.4(c), there is $b_{\nu 1} \in K_v$ such that

$\Phi(b_{\nu 1}) = \frac{a_{i\nu} - a'_\nu}{c}$ and $\Phi'(b_{\nu 1}) \neq 0$. Set $b_{\nu 2} = b_{\nu 3} = 0$. By assumption, $\mathbf{a}_i \in V(K_x)$. Hence, $(\mathbf{a}_i, \mathbf{b}) \in W(K_x)$. Moreover, $(\mathbf{a}_i, \mathbf{b})$ is a simple point on W: the Jacobi matrix of (6) at this point with respect to the variables

$$Z_1, \ldots, Z_n, Y_{11}, \ldots, Y_{n1}, Y_{21}, \ldots, Y_{n2}, Y_{31}, \ldots, Y_{n3}$$

is the block matrix $J = \begin{pmatrix} J_1 & 0 & 0 & 0 \\ * & J_2 & * & * \end{pmatrix}$ of order $(m+n) \times (n+n+n+n)$ where $J_1 = \left(\frac{\partial f_\mu}{\partial Z_j}(\mathbf{a}_i)\right)$, and $J_2 = -\mathrm{diag}(\Phi'(b_{11}), \ldots, \Phi'(b_{n1}))$. Since V is smooth, $\mathrm{rank}(J_1) = n - \mathrm{div}(V)$. Since $\Phi'(b_{\nu 1}) \neq 0$, $\mathrm{rank}(J_2) = n$. Hence, $\mathrm{rank}(J) = n - \dim(V) + n = 4n - (\dim(V) + 2n) = 4n - \dim(W)$, so $(\mathbf{a}_i, \mathbf{b}) \in W_{\mathrm{simp}}(K_x)$.

By assumption, K is P$\mathcal{X}$C. Hence, (6) has a solution $(\mathbf{a}, \mathbf{b})$ in $W(K)$. The first m equations of (6) ensure that $\mathbf{a} \in V$; the other n equations imply, by Lemma 3.4(a), that $v_x(\frac{a_\nu - a'_\nu}{c}) \geq 0$, for all $x \in X$. ∎

5. Local Preparations

The block approximation theorem is proved in Section 4 in the setup of proper field-valuation structures of valued Henselian fields with bounded residue fields. We proceed to prove the block approximation theorem for P-adically closed fields. In this case, all technical results which are needed in the setup of field-valuation structures are shown to follow from basic natural assumptions. One of the most difficult ones is the continuity of the maps $\lambda_L\colon X_L \to \mathrm{Val}(L)$ (Condition (3d) of Section 4). In this section we make local preparation for the proof of this fact. The conclusion of the proof follows in the next section.

Lemma 5.1: *Let (K,v) be a valued field, K_v a Henselian closure, and L a finite separable extension of K in K_v. Then v has an open neighborhood $\mathbf{U}$ in $\mathrm{Val}(K)$ satisfying this: For each $w \in \mathbf{U}$ there is a K-embedding of L in a Henselian closure of (K,w).*

Proof: Lemma 8.3 of [HJP07] gives a primitive element x for L/K such that $f(X) = \mathrm{irr}(x,K) = X^n + X^{n-1} + a_{n-2}X^{n-2} + \cdots + a_0$ with $v(a_i) > 0$, $i = 0,\ldots,n-2$. Then $\mathbf{U} = \{w \in \mathrm{Val}(K) \mid\ w(a_i) > 0,\ i = 0,\ldots,n-2\}$ is an open neighborhood of v in $\mathrm{Val}(K)$. Now we apply [HJP07, Lemma 8.3] in the other direction to conclude: For each $w \in \mathbf{U}$ there is a K-embedding of L in a Henselian closure of (K,w). ∎

Corollary 5.2: *Let K be a field and $\mathbf{B}$ a closed subset of $\mathrm{Val}(K)$. Denote the set of all Henselian closures K_v of K inside K_s at valuations $v \in \mathbf{B}$ by $\mathcal{X}$. Suppose the residue field of each $v \in \mathbf{B}$ is finite and $\mathcal{X}$ is étale profinite. Then the map $\chi\colon \mathcal{X} \to \mathbf{B}$ given by $K_v \mapsto v$ is étale continuous and open.*

Proof: By Lemma 2.4, each field in $\mathcal{X}$ is the Henselian closure of K at a unique valuation belonging to $\mathbf{B}$, so χ is well defined. The set $\mathcal{X}$ is closed under Galois conjugation. Since $\mathcal{X}$ is étale profinite, so is the quotient space, $\mathcal{X}/\mathrm{Gal}(K)$. Moreover, the quotient map $\pi\colon \mathcal{X} \to \mathcal{X}/\mathrm{Gal}(K)$ is continuous and open [HaJ85, Claim 1.6].

By Lemma 5.1, the map $\beta\colon \mathbf{B} \to \mathcal{X}/\mathrm{Gal}(K)$ which maps each $v \in \mathbf{B}$ onto the class of K_v is étale continuous. In addition, β bijective. Since $\mathbf{B}$ is compact (Definition 3.6) and $\mathcal{X}/\mathrm{Gal}(K)$ is Hausdorff (because $\mathcal{X}$ is étale profinite), β is a homeomorphism.

Finally we observe that $\chi = \beta^{-1} \circ \pi$ to conclude that χ is continuous and open. ∎

LEMMA 5.3: *Let K be a field, S a finite set of prime numbers, and m a positive integer. Denote the set of all $v \in \mathrm{Val}(K)$ with $\mathrm{char}(\bar{K}_v) \in S$ and $|\bar{K}_v| \leq m$ by $\mathbf{B}$. Then $\mathbf{B}$ is closed in $\mathrm{Val}(K)$.*

Proof: For each $p \in S$ let $\mathbf{B}_p = \{v \in \mathbf{B} \mid \mathrm{char}(\bar{K}_v) = p\}$. Then $\mathbf{B} = \dot{\bigcup}_{p \in S} \mathbf{B}_p$. It suffices to prove each $\mathbf{B}_p$ is closed. So, assume S consists of a single prime number p.

We consider w in the closure of $\mathbf{B}$ in $\mathrm{Val}(K)$, let $p' = \mathrm{char}(\bar{K}_w)$, and assume $p' \neq p$. Then $w(p') > 0$, that is the set $\{v \in \mathrm{Val}(K) \mid v(p') > 0\}$ is an open neighborhood of w in $\mathrm{Val}(K)$ (Definition 3.6). Hence, there is a $v \in \mathbf{B}$ with $v(p') > 0$. This contradiction to $\mathrm{char}(\bar{K}_v) = p$ proves that $p' = p$.

Now assume $|\bar{K}_w| > m$. Then there are $a_1, \ldots, a_{m+1} \in K$ whose reductions in $\bar{K}_w$ are distinct. In other words, $w(a_i - a_j) = 0$ for $i \neq j$, hence there exists $v \in \mathbf{B}$ with $v(a_i - a_j) = 0$ for $i \neq j$. This contradiction to $|\bar{K}_v| \leq m$ proves $|\bar{K}_w| \leq m$. ∎

Let (F, v) be a valued field. We call (F, v) P**-adic** if there is a prime number p satisfying these conditions:

(1a) The residue field $\bar{F}_v$ is finite, say with $q = p^f$ elements.

(1b) There is a $\pi \in F$ with a smallest positive value $v(\pi)$ in $v(F^\times)$. Thus, $\mathfrak{m}_v = \pi O_v$. We call π a **prime element** of (F, v).

(1c) There is a positive integer e with $v(p) = ev(\pi)$.

We call (e, q, f) the **type** of (F, v) and say (F, v) is P**-adically closed** if (F, v) admits no finite proper P-adically closed extension of the same type [HJP05, §8].

LEMMA 5.4: *Let (F, v) be a P-adically closed field satisfying (1) and w valuation of F which is strictly coarser than v. Let $\tilde{w}$ be an extension of w to a valuation of $\tilde{F}$. Then $\tilde{w}$ is unramified over F and its decomposition group over F is $\mathrm{Gal}(F)$. In particular, the homomorphism $\mathrm{Gal}(F) \to \mathrm{Gal}(\bar{F}_w)$ is bijective.*

Proof: Let $\bar{v}$ be the valuation of $\bar{F}_w$ induced by v. We denote reduction of elements of O_w modulo $\mathfrak{m}_w$ by a bar. We note that $\pi \in U_w$, otherwise $\mathfrak{m}_v = \pi O_v \subseteq \mathfrak{m}_w$, hence

$\mathfrak{m}_v = \mathfrak{m}_w$, so $O_v = O_w$, contradicting our assumption. By [Jar91b, §3], $\Gamma_{\bar{v}}$ is a convex subgroup of Γ_v that contains $v(\pi) = \bar{v}(\bar{\pi})$. For each positive integer n and each $\gamma \in \Gamma_v$ there are $k \in \mathbb{Z}$ and $\delta \in \Gamma_v$ with $\gamma = kv(\pi) + n\delta$ [HJP05, Prop. 8.2(g)]. Therefore, $\Gamma_w = \Gamma_v/\Gamma_{\bar{v}}$ is divisible.

$$\begin{tikzcd}
F \arrow[d, dash] & & \\
O_w \arrow[r] \arrow[d, dash] & \bar{F}_w \arrow[d, dash] & \\
O_v \arrow[r] \arrow[d, dash] & O_{\bar{v}} \arrow[r] \arrow[d, dash] & \bar{F}_v = \bar{F}_{\bar{v}} \\
\mathfrak{m}_v = \pi O_v \arrow[r] \arrow[d, dash] & \mathfrak{m}_{\bar{v}} & \\
\mathfrak{m}_w & &
\end{tikzcd}
\qquad
\begin{tikzcd}
F^\times \arrow[r] \arrow[d, dash] & \Gamma_v \arrow[r] \arrow[d, dash] & \Gamma_w \arrow[d, dash] \\
U_w \arrow[r] \arrow[d, dash] & \Gamma_{\bar{v}} \arrow[r] \arrow[d, dash] & 1 \\
U_v \arrow[r] & 1 & \\
& &
\end{tikzcd}$$

Since (F, v) is Henselian [HJP05, Prop. 8.2(g)], so is (F, w) [Jar91b, Prop. 13.1]. By assumption, the residue field $\bar{K}_v$ of $(\bar{F}_w, \bar{v})$ has characteristic p and $p = u\pi^e$ with $v(u) = 0$. Hence, $p = \bar{u}\bar{\pi}^e \neq 0$, so $\mathrm{char}(\bar{F}_w) = 0$. Therefore, the formula $[F' : F] = e(F'/F)f(F'/F)$ holds for each finite extension F' of F with respect to the unique extension of w to F' [Rbn64, p. 236]. Since Γ_w is divisible, $e(F'/F) = 1$. Hence, w is unramified in F' and $[\bar{F}'_w : \bar{F}_w] = [F' : F]$. It follows that the decomposition group of $\tilde{w}$ over F is $\mathrm{Gal}(F)$. ∎

Lemma 5.5: *Let (F, v) be a* P*-adically closed field and v' a valuation of F. Then either $F_{v'} = F$ or $F_{v'} = \tilde{F}$. If $\bar{F}_{v'}$ is finite, then $F_{v'} = F$ and $v' = v$.*

Proof: Set $L = F_{v'}$ and let v_L and v'_L be extensions of v and v' to L. Then both v_L and v'_L are Henselian.

First suppose v_L and v'_L are incomparable. Then, L has a valuation w_L which is coarser than both v_L and v'_L such that $\bar{L}_{w_L}$ is separably closed [Jar91b, Prop. 13.4]. In particular, w_L is strictly coarser that v_L. Denote the restriction of w_L to F by w. Then w is strictly coarser than v [Jar91b, Cor. 6.6]. By Lemma 5.4, $\mathrm{Gal}(L) \cong \mathrm{Gal}(\bar{L}_{w_L}) = 1$. Hence, $L = \tilde{F}$.

Now suppose v_L and v'_L are comparable. Then v and v' are comparable. If v were strictly coarser than v', then $\bar{F}_{v'}$ would be a residue field of a nontrivial valuation of

$\bar{F}_v$. Since $\bar{F}_v$ is finite, this is a contradiction. Hence, $v \prec v'$, so $F_{v'}$ can be F-embedded in $F_v = F$ [Jar, Cor. 14.4]. Consequently, $F_{v'} = F$.

Finally, if $\bar{F}_{v'}$ is finite, then $F_{v'} \neq \tilde{F}$. Hence, by the preceding two paragraphs, v and v' are comparable. Therefore, one of the fields $\bar{F}_v$ and $\bar{F}_{v'}$ is a residue field of the other. Since both fields are finite, this implies $v' = v$. ∎

PROPOSITION 5.6: *Let $\mathcal{X}$ be a nonempty family of* P*-adically closed algebraic extensions of a field K. Suppose $\mathcal{X}$ is étale compact and closed under elementary equivalence (i.e. $F \in \mathcal{X}$, $F' \in \mathrm{AlgExt}(K)$, and $F' \equiv F$ imply $F' \in \mathcal{X}$). Suppose also K is P$\mathcal{X}$C. Let w be a valuation of K with $K_w \neq \tilde{K}$. Then $K_w \in \mathcal{X}$.*

Proof: Lemma 2.2 gives an $F \in \mathcal{X}$ with $F \subseteq K_{w,\mathrm{alg}}$. Then $F \subseteq K_w$. By Lemma 5.5, $K_w = F$. ∎

Notation 5.7: For each valued field (K, v) let $\psi_v \colon K \to \bar{K}_v \cup \{\infty\}$ be the place extending the residue map $O_v \to \bar{K}_v$. If w is a coarser valuation of K than v and $\bar{v}$ is the unique valuation of $\bar{K}_w$ with $\psi_w^{-1}(O_{\bar{v}}) = O_v$ [Jar91b, §3], we write $\psi_v = \psi_{\bar{v}} \circ \psi_w$ and note that $\psi_v(x) = \psi_{\bar{v}}(\psi_w(x))$ for all $x \in K$ if we set $\psi_{\bar{v}}(\infty) = \infty$. ∎

The next result generalizes [HaJ88, Lemma 6.7].

LEMMA 5.8: *Let $\mathbb{F}$ be a finite extension of $\mathbb{Q}_p$ and $\bar{v}$ the* P*-adic valuation of $\mathbb{F}$. Let F be a field elementarily equivalent to $\mathbb{F}$ and v the corresponding* P*-adic valuation [HJP05, Prop. 8.2(h)]. Then there is a $\dot{v} \in \mathrm{Val}(F)$ (possibly trivial) coarser than v with $\bar{F}_{\dot{v}} \subseteq \mathbb{F}$ and $\psi_v = \psi_{\bar{v}} \circ \psi_{\dot{v}}$ (Diagram (2)). Moreover, $\bar{F}_{\dot{v}}$ is a* P*-adically closed field, elementarily equivalent to $\mathbb{F}$, the restriction of $\bar{v}$ to $\bar{F}_{\dot{v}}$ is its* P*-adic valuation, and it is discrete. Finally, if (F, v) is a* P*-adic closure of (K, v), then $(\bar{F}_{\dot{v}}, \bar{v})$ is a* P*-adic closure of $(\bar{K}_{\dot{v}}, \bar{v})$.*

Proof: Let $F_0 = F \cap \tilde{\mathbb{Q}}$ and $\mathbb{F}_0 = \mathbb{F} \cap \tilde{\mathbb{Q}}$. By [HJP05, Prop. 8.2(f)], $F_0 \equiv F \equiv \mathbb{F} \equiv \mathbb{F}_0$. Hence, $F_0 \cong \mathbb{F}_0$ [FrJ05, Cor. 20.6.4(b)]. Without loss identify F_0 with $\mathbb{F}_0$. Again, by [HJP05, Prop. 8.2(b),(f)], F_0 admits a unique P-adically closed valuation v_0 which is the restriction of both $\bar{v}$ and v. Moreover, $\bar{F}_v = \bar{F}_{0,v_0} = \bar{\mathbb{F}}_{\bar{v}}$ and any prime element π of (F_0, v_0) is also a prime element of both $(\mathbb{F}, \bar{v})$ and (F, v).

To construct $\dot{v}$, we choose a system of representatives R for $\bar{F}_v$ in O_{v_0}. Then

for each element $a \in O_v$ there are unique $a_0 \in R$ and $b_1 \in O_v$ with $a = a_0 + b_1\pi$. Similarly there are unique $a_1 \in R$ and $b_2 \in O_v$ with $b_1 = a_1 + b_2\pi$. Thus, $a = a_0 + a_1\pi + b_2\pi^2$. If we continue by induction, we find unique $a_0, a_1, a_2, \ldots$ in R with $a \equiv \sum_{i=0}^{n} a_i\pi^i \mod \pi^{n+1}O_v, \quad n \in \mathbb{N}$. The infinite series $\sum_{i=0}^{\infty} a_i\pi^i$ converges to an element $\psi(a) \in \mathbb{F}$. Similarly, each $a \in \mathbb{F}$ has a unique representation as $a = \sum_{i=0}^{\infty} a_i\pi^i$ with $a_i \in R$ for all i.

(2)

$$\begin{tikzcd}
 & \mathbb{F} \cup \{\infty\} \arrow[dd, dash] \arrow[ddr, "\psi_{\bar{v}}"] & \\
F \arrow[r, "\psi_{\dot{v}}"] \arrow[drr, "\psi_v"] \arrow[dd, dash] & \bar{F}_{\dot{v}} \cup \{\infty\} \arrow[dd, dash] & \\
 & & \\
O_{\dot{v}} \arrow[r] \arrow[dd, dash] & \bar{F}_{\dot{v}} \arrow[r] \arrow[dd, dash] & \bar{F}_v \cup \{\infty\} \arrow[dd, dash] \\
 & & \\
O_v \arrow[r, "\psi"] \arrow[dd, dash] & O_{\bar{v}} \arrow[r] \arrow[dd, dash] & \bar{\mathbb{F}}_v \\
 & & \\
\mathfrak{m}_v \arrow[r] & \mathfrak{m}_{\bar{v}} &
\end{tikzcd}$$

This gives a homomorphism $\psi\colon O_v \to \mathbb{F}$ with $\mathrm{Ker}(\psi) = \bigcap_{i=1}^{\infty} \pi^i O_v$ that maps $O_v \cap F_0$ identically onto itself, in particular $\psi(\pi) = \pi$. The local ring of O_v at $\mathrm{Ker}(\psi)$ is a valuation ring of some $\dot{v} \in \mathrm{Val}(F)$ with residue field $\bar{F}_{\dot{v}} \subseteq \mathbb{F}$ and $\psi_v = \psi_{\bar{v}} \circ \psi_{\dot{v}}$. Note that $\mathrm{Ker}(\psi) \neq \pi O_v = \mathfrak{m}_v$, so $\dot{v}$ is strictly coarser than v. But it may happen that $\mathrm{Ker}(\psi) = 0$. In this case $O_{\dot{v}} = F$ and $\dot{v}$ is trivial.

The P-adic valuation $\bar{v}$ of $\mathbb{F}$ is discrete. Hence, so is its restriction to $\bar{F}_{\dot{v}}$ (which we also denote by $\bar{v}$). Then, the residue field of $(\bar{F}_{\dot{v}}, \bar{v})$ is $\bar{F}_v$ and $\bar{v}(\pi) = v(\pi)$ is the smallest positive value in $\bar{v}(\bar{F}_{\dot{v}})$. Since (F, v) is Henselian, so is $(\bar{F}_{\dot{v}}, \bar{v})$ [Jar91b, Prop. 13.1]. By [HJP05, Prop. 8.2(g)], $\bar{F}_{\dot{v}}$ is P-adically closed and $\bar{v}$ is its P-adic valuation. Since $F_0 \subseteq \bar{F}_{\dot{v}} \subseteq \mathbb{F}$, we have $F_0 = \bar{F}_{\dot{v}} \cap \tilde{\mathbb{Q}}$. We conclude from [HJP05, Prop. 8.2(f)] that $\bar{F}_{\dot{v}} \equiv F_0 \equiv \mathbb{F}$.

Finally, suppose (F, v) is a P-closure of a P-adic field (K, v). Then K contains a prime element π' for (F, v). Its image $\bar{\pi}'$ in $\bar{F}_{\dot{v}}$ is a prime element for $(\bar{K}_{\dot{v}}, \bar{v})$. Also,

the residue field of $(\bar{K}_{\dot{v}}, \bar{v})$ is $\bar{K}_v$, which is $\bar{F}_v$. Therefore, $(\bar{F}_{\dot{v}}, \bar{v})$ is a P-adic closure of $(\bar{K}_{\dot{v}}, \bar{v})$. ∎

PROPOSITION 5.9: *Let v be a* P*-adic valuation of a field K and F, F'* P*-adic closures of (K, v). Then $F \equiv F'$.*

Proof: If v is discrete, then $F \cong_K F'$ [HJP05, Prop. 8.2(d)], so $F \equiv F'$.

Suppose v is not discrete. Let p be the residue characteristic of (K, v). By [HJP05, Prop. 8.2(j)], F (resp. F') is elementarily equivalent to a finite extension $\mathbb{F}$ (resp. $\mathbb{F}'$) of $\mathbb{Q}_p$. Let v_F (resp. $v_{F'}$) be the unique P-adic valuation of F (resp. F') extending v [HJP05, Prop. 8.2(c),(d)]. Lemma 5.8 gives a valuation $\dot{v}_F$ (resp. $\dot{v}_{F'}$) with residue field $\bar{F}_{\dot{v}} \subseteq \mathbb{F}$ (resp. $\bar{\mathbb{F}}_{\dot{v}'} \subseteq \mathbb{F}'$). Let $\dot{v}$ (resp. $\dot{v}'$) be the restriction of $\dot{v}_F$ (resp. $\dot{v}_{F'}$) to K. Then both $\dot{v}$ and $\dot{v}'$ are strictly coarser than v. Hence, one of them is coarser than the other, say $v \prec \dot{v} \prec \dot{v}'$. By Lemma 5.8, the residue valuation $\dot{v}'_F/v_F$ of $\bar{F}_{\dot{v}'}$ is discrete. Hence, $\dot{v}'/v$ is discrete. Therefore, $\dot{v} = \dot{v}'$.

It follows that the residue valuations of $\bar{F}_{\dot{v}}$ and $\bar{F}_{\dot{v}'}$ coincide on $\bar{K}_{\dot{v}}$. Denote their common restriction to $\bar{K}_{\dot{v}}$ by $\bar{v}$. It is discrete and both $\bar{F}_{\dot{v}}$ and $\bar{F}_{\dot{v}'}$ are P-adic closures of $(\bar{K}_{\dot{v}}, \bar{v})$ (Lemma 5.8). By [HJP05, Prop. 8.2(d)], $\bar{F}_{\dot{v}} \cong_{\bar{K}_v} \bar{F}_{\dot{v}'}$. Hence, by Lemma 5.8, $F \equiv \bar{F}_{\dot{v}} \equiv \bar{F}_{\dot{v}'} \equiv F'$. ∎

Notice that $\mathbb{Q}$ is p-adically dense in $\mathbb{Q}_p$, so $\mathbb{Q}_{p,\mathrm{alg}} = \mathbb{Q}_p \cap \tilde{\mathbb{Q}}$.

LEMMA 5.10: *Let p be a prime number, $\sigma \in \mathrm{Gal}(\mathbb{Q})$, M an algebraic extension of $\mathbb{Q}$, and M' a finite extension of M not equal to $\tilde{\mathbb{Q}}$. Suppose $\mathbb{Q}_{p,\mathrm{alg}} \subseteq M$ and $\mathbb{Q}^{\sigma}_{p,\mathrm{alg}} \subseteq M'$. Then $\mathbb{Q}^{\sigma}_{p,\mathrm{alg}} = \mathbb{Q}_{p,\mathrm{alg}}$.*

Proof: The field $\mathbb{Q}$ is p-adically dense in both $\mathbb{Q}_{p,\mathrm{alg}}$ and $\mathbb{Q}^{\sigma}_{p,\mathrm{alg}}$. If $\mathbb{Q}^{\sigma}_{p,\mathrm{alg}} \neq \mathbb{Q}_{p,\mathrm{alg}}$, then by Proposition 1.11, $\mathbb{Q}_{p,\mathrm{alg}}\mathbb{Q}^{\sigma}_{p,\mathrm{alg}} = \tilde{\mathbb{Q}}$. This contradicts the fact that the left hand side is contained in M' and $M' \neq \tilde{\mathbb{Q}}$. Consequently, $\mathbb{Q}^{\sigma}_{p,\mathrm{alg}} = \mathbb{Q}_{p,\mathrm{alg}}$. ∎

6. Continuity

We apply the results of Section 5 to prove the continuity of the maps λ_L under appropriate assumptions.

Data 6.1: S-fields. Let S be a finite set of prime numbers. For each $p \in S$ let $\mathcal{F}_p$ be a finite set of finite extensions of $\mathbb{Q}_p$. Put $\mathcal{F} = \bigcup_{p\in S} \mathcal{F}_p$. Suppose $\mathcal{F}$ is **closed under Galois-isomorphism**; that is if $\mathbb{F}$ is a finite extension of $\mathbb{Q}_l$ and $\mathbb{F}'$ is a finite extension of $\mathbb{Q}_{l'}$ for some prime numbers l, l', $\mathbb{F} \in \mathcal{F}$, and $\mathrm{Gal}(\mathbb{F}') \cong \mathrm{Gal}(\mathbb{F})$, then $\mathbb{F}' \in \mathcal{F}$.

Let K be a field. For each finite extension $\mathbb{F}$ of $\mathbb{Q}_p$ let $\mathrm{AlgExt}(K, \mathbb{F})$ be the set of all algebraic extensions of K which are elementarily equivalent to $\mathbb{F}$. Then let $\mathbf{B}_{K,\mathbb{F}}$ be the set of all P-adic valuations v of K such that (K, v) has a P-adic closure (F, w) with $F \equiv \mathbb{F}$. If $\mathbb{F}'$ is a finite extension of $\mathbb{Q}_{p'}$ and $\mathbb{F}' \not\equiv \mathbb{F}$, then $\mathbf{B}_{K,\mathbb{F}} \bigcap \mathbf{B}_{K,\mathbb{F}'} = \emptyset$ (Proposition 5.9).

We set $\mathrm{AlgExt}(K, \mathcal{F}) = \bigcup_{\mathbb{F}\in\mathcal{F}} \mathrm{AlgExt}(K, \mathbb{F})$ and $\mathbf{B}_{K,\mathcal{F}} = \bigcup_{\mathbb{F}\in\mathcal{F}} \mathbf{B}_{K,\mathbf{B}}$.

For each subset $\mathcal{Y}$ of $\mathrm{AlgExt}(K)$ let $\mathcal{Y}_{\min}$ be the set of all minimal elements of $\mathcal{Y}$ with respect to inclusion. If $\mathcal{Y}$ is closed under conjugation with elements of $\mathrm{Gal}(K)$, then so is $\mathcal{Y}_{\min}$. This is the case for $\mathrm{AlgExt}(K, \mathbb{F})$, hence also for $\mathrm{AlgExt}(K, \mathcal{F})$.

Let $\mathcal{K}$ be a family of algebraic extensions of K. We say K is **pseudo-$\mathcal{K}$-closed** (abbreviated P$\mathcal{K}$C) if every variety defined over K with a simple F-rational point for each $F \in \mathcal{K}$ has a K-rational point. In that case K is also P$\mathcal{K}'$C for each family $\mathcal{K}'$ of algebraic extensions of K that contains $\mathcal{K}$. We say K is P$\mathcal{F}$C if K is pseudo-$\mathrm{AlgExt}(K, \mathcal{F})$-closed. ∎

Lemma 6.2: *Let K, S, and $\mathcal{F}$ be as in Data 6.1.*

(a) $\mathrm{AlgExt}(K, \mathcal{F})$ *is strictly closed and étale compact.*

(b) *Suppose K is P$\mathcal{F}$C. Then* $\mathrm{AlgExt}(K, \mathcal{F})_{\min}$ *is étale profinite.*

(c) *Suppose K is* P$\mathcal{F}$C. *Then* $\mathbf{B}_{K,\mathcal{F}}$ *is closed in* $\mathrm{Val}(K)$.

Proof of (a): By [HJP05, Lemma 10.1], each of the sets $\mathrm{AlgExt}(K, \mathbb{F})$ is strictly closed in $\mathrm{AlgExt}(K)$. Hence, $\mathrm{AlgExt}(K, \mathcal{F}) = \bigcup_{\mathbb{F}\in\mathcal{F}} \mathrm{AlgExt}(K, \mathbb{F})$ is strictly closed. By [HJP07, Remark 1.2], $\mathrm{AlgExt}(K, \mathcal{F})$ is étale compact.

Proof of (b): Let $\mathcal{G} = \bigcup_{\mathbb{F}\in\mathcal{F}} \{\mathrm{Gal}(F) \mid F \in \mathrm{AlgExt}(K) \text{ and } \mathrm{Gal}(F) \cong \mathrm{Gal}(\mathbb{F})\}$. By

assumption, $\mathcal{F}$ is closed under Galois equivalence. Hence, by [HJP05, Thm. 10.4], $(\mathrm{Gal}(K), \mathcal{G}_{\max})$ is a proper group structure. In particular, $\mathcal{G}_{\max}$ is étale profinite [HJP05, Definition preceding Prop. 6.3]. By [HJP05, Lemma 10.3],

$$\mathcal{G} = \bigcup_{\mathbb{F}\in\mathcal{F}} \{\mathrm{Gal}(F) \mid F \in \mathrm{AlgExt}(K) \text{ and } F \equiv \mathbb{F}\} = \{\mathrm{Gal}(F) \mid F \in \mathrm{AlgExt}(K, \mathcal{F}\}.$$

In the terminology of fields, this means that $\mathrm{AlgExt}(K, \mathcal{F})_{\min}$ is étale profinite.

Proof of (c): Set $\mathbf{B} = \mathbf{B}_{K,\mathcal{F}}$. Since $\mathcal{F}$ is finite, there is a positive integer m with $|\bar{K}_v| \le m$ for all $v \in \mathbf{B}$. Let $\mathbf{B}' = \{v \in \mathrm{Val}(K) \mid |\bar{K}_v| \le m\}$. Then $\mathbf{B} \subseteq \mathbf{B}'$. By Lemma 5.3, $\mathbf{B}'$ is closed in $\mathrm{Val}(K)$.

Consider w in the closure of $\mathbf{B}$. Then $w \in \mathbf{B}'$, so $|\bar{K}_w| \le m$. In particular, $K_w \ne \tilde{K}$. By (a), $\mathcal{X} = \mathrm{AlgExt}(K, \mathcal{F})$ is étale compact. In addition, $\mathcal{X}$ is closed under elementary equivalence and K is P$\mathcal{X}$C. By Proposition 5.6, $K_w \in \mathcal{X}$. In particular, K_w is a P-adic closure of some $v \in \mathbf{B}$. Let v' be the corresponding extension of v to K_w. Then the residue field of K_w at v' is finite. Let w^h be the Henselian valuation of K_w lying over w. By Lemma 5.5, $w^h = v'$. Consequently, $w = v \in \mathbf{B}$. ∎

Lemma 6.3: *Let S, $\mathcal{F}$, and K be as in Data 6.1. Suppose K is P$\mathcal{F}$C. Let $\mathcal{X} = \mathrm{AlgExt}(K, \mathcal{F})_{\min}$. Then:*

(a) *Each $F \in \mathrm{AlgExt}(K, \mathcal{F})$ admits a unique* P*-adic valuation w_F; moreover, (F, w_F) is* P*-adically closed.*

(b) *$\{(F, w_F) \mid F \in \mathcal{X}\}$ is the set of all Henselian closures of K at valuations $v \in \mathbf{B}_{K,\mathcal{F}}$.*

(c) *Let L be a finite extension of K. Then the map*

$$\lambda_L \colon \mathrm{AlgExt}(L) \cap \mathcal{X} \to \mathrm{Val}(L)$$

given by $F \mapsto w_F|_L$ is étale continuous. Moreover, $\lambda_K \colon \mathcal{X} \to \mathbf{B}_{K,\mathcal{F}}$ is an open surjection.

Proof of (a): Let $F \in \mathrm{AlgExt}(K, \mathcal{F})$. Then $F \equiv \mathbb{F}$ for some $\mathbb{F} \in \mathcal{F}$. By [HJP05, Prop. 8.2(h)], F admits a P-adic valuation w_F such that (F, w_F) is P-adically closed. Let w be another P-adic valuation on F. Let (F', w') be a P-adic closure of (F, w) and

extend w_F to a valuation w'_F on F'. Since P-adically closed fields are Henselian, and algebraic extensions of Henselian fields are Henselian, F' is Henselian with respect to both w' and w'_F. Moreover, the residue field $(\overline{F'})_{w'}$ is finite and $(\overline{F'})_{w'_F}$ is an algebraic extension of a finite field. By Lemma 2.4, $w'_F = w'$. Restriction to F gives $w_F = w$.

Proof of (b): By Lemma 6.2(a), $\mathrm{AlgExt}(K, \mathcal{F})$ is étale compact.

Let $F \in \mathcal{X}$. Put $w = w_F$ and $v = w|_K$. Then $v \in \mathbf{B}_{K,\mathcal{F}}$. By (a), (F, w) is P-adically closed, hence Henselian. By assumption, K is P$\mathcal{F}$C. Hence, by Proposition 2.3(a), (F, w) is a Henselian closure of (K, v).

Conversely, let (F, w) be a Henselian closure of (K, v), with $v \in \mathbf{B}_{K,\mathcal{F}}$. Then w is a P-adic valuation of F of the same type as v. Let (F', w') be a P-adic closure of (F, w). Then (F', w') is also a P-adic closure of (K, v). By the definition of $\mathbf{B}_{K,\mathcal{F}}$, (K, v) has a P-adic closure (K', v') with $K' \equiv \mathbb{F}$ for some $\mathbb{F} \in \mathcal{F}$. By Proposition 5.9, $F' \equiv K'$, hence $F' \equiv \mathbb{F}$, so $F' \in \mathrm{AlgExt}(K, \mathcal{F})$. By [HJP05, Lemma 2.6], F' contains a minimal element E of $\mathrm{AlgExt}(K, \mathcal{F})$; that is, $E \in \mathcal{X}$. Then $w_0 = w'|_E$ is a P-adic valuation of E of the same type as w and v and $w_0|_K = v$. By the preceding paragraph, (E, w_0) is a Henselian closure of (K, v) and P-adically closed. The latter gives $E = F'$, so $F \subseteq E$, the former gives that $F = E \in \mathcal{X}$. By (a), $w = w_F$.

Proof of (c): First assume $L = K$. Then $\lambda_K(\mathcal{X}) = \mathbf{B}_{K,\mathcal{F}}$. Indeed, let $F \in \mathcal{X}$. By definition, $\lambda_K(F) = w_F|_K \in \mathbf{B}_{K,\mathcal{F}}$. Conversely, let $v \in \mathbf{B}_{K,\mathcal{F}}$. Let (F, w) be a Henselian closure of (K, v). By (b), $F \in \mathcal{X}$. By (a), $w = w_F$. Hence, $\lambda_K(F) = v$.

By Lemma 6.2, $\mathcal{X}$ is étale profinite and $\mathbf{B}_{K,\mathcal{F}}$ is closed in $\mathrm{Val}(K)$. The residue field of K at each $v \in \mathcal{B}_{K,\mathcal{F}}$ is finite. Hence, by Corollary 5.2, $\lambda_K \colon \mathcal{X} \to \mathbf{B}_{K,\mathcal{F}}$ is étale continuous and open.

Now let L be an arbitrary finite extension of K. We denote the set of all finite extensions of $\mathbb{Q}_p$, with p ranging over all prime numbers, by $\mathcal{P}$. Let $\mathcal{F}_L$ be the set of all $\mathbb{F}' \in \mathcal{P}$ that are elementarily equivalent to FL for some $F \in \mathrm{AlgExt}(K, \mathcal{F})$. We claim that $\mathcal{F}_L$ is finite.

Indeed, consider $F \in \mathrm{AlgExt}(K, \mathcal{F})$. Then F is a P-adically closed field, elementarily equivalent to a finite extension $\mathbb{F}$ of $\mathbb{Q}_p$ for some $p \in S$. By [FrJ05, Cor. 20.6.4(b)],

there is an isomorphism $\sigma\colon \mathbb{F}\cap\tilde{\mathbb{Q}} \to F\cap\tilde{\mathbb{Q}}$. In particular, $(\mathbb{Q}_p\cap\tilde{\mathbb{Q}})^\sigma \subseteq F\cap\tilde{\mathbb{Q}}$. By [HJP05, Prop. 8.2(i)], FL is elementarily equivalent to a finite extension $\mathbb{F}'$ of $\mathbb{Q}_p$. Again, by [FrJ05, Cor. 20.6.4(b)], there is an isomorphism $\tau\colon \mathbb{F}'\cap\tilde{\mathbb{Q}} \to FL\cap\tilde{\mathbb{Q}}$. In particular, $(\mathbb{Q}_p\cap\tilde{\mathbb{Q}})^\tau \subseteq FL\cap\tilde{\mathbb{Q}}$. Since $FL\cap\tilde{\mathbb{Q}}$ is a finite extension of $F\cap\tilde{\mathbb{Q}}$, Lemma 5.10 implies that $(\mathbb{Q}_p\cap\tilde{\mathbb{Q}})^\sigma = (\mathbb{Q}_p\cap\tilde{\mathbb{Q}})^\tau$. Hence, by [HJP05, Prop. 8.2(l)],

$$
\begin{aligned}
[\mathbb{F}' : \mathbb{Q}_p] &= [\mathbb{F}'\cap\tilde{\mathbb{Q}} : \mathbb{Q}_p\cap\tilde{\mathbb{Q}}] \\
&= [FL\cap\tilde{\mathbb{Q}} : (\mathbb{Q}_p\cap\tilde{\mathbb{Q}})^\sigma] \\
&= [FL\cap\tilde{\mathbb{Q}} : F\cap\tilde{\mathbb{Q}}][F\cap\tilde{\mathbb{Q}} : (\mathbb{Q}_p\cap\tilde{\mathbb{Q}})^\sigma] \\
&= [FL\cap\tilde{\mathbb{Q}} : F\cap\tilde{\mathbb{Q}}][\mathbb{F}\cap\tilde{\mathbb{Q}} : \mathbb{Q}_p\cap\tilde{\mathbb{Q}}] \\
&= [FL : F][\mathbb{F} : \mathbb{Q}_p] \le [L : K][\mathbb{F} : \mathbb{Q}_p].
\end{aligned} \tag{2}
$$

Since $\mathcal{F}$ is a finite set, the right hand side of (2) is bounded as p ranges on S and $\mathbb{F}$ ranges on $\mathcal{F}$. Hence, by [HJP05, Prop. 8.2(k)], there are only finitely many possibilities for $\mathbb{F}'$.

If $F \in \mathrm{AlgExt}(K,\mathcal{F})$, then there exists $\mathbb{F} \in \mathcal{F}$ with $F \equiv \mathbb{F}$. Since FL is a finite extension of F, it is elementarily equivalent to a finite extension $\mathbb{F}'$ of $\mathbb{F}$. Thus, $\mathbb{F}' \in \mathcal{F}_L$. Hence, $FL \in \mathrm{AlgExt}(L,\mathcal{F}_L)$. Therefore, $\mathrm{AlgExt}(K,\mathcal{F})L \subseteq \mathrm{AlgExt}(L,\mathcal{F}_L)$. Since K is pseudo-$\mathrm{AlgExt}(K,\mathcal{F})$-closed, L is pseudo-$\mathrm{AlgExt}(K,\mathcal{F})L$-closed [Jar91a, Lemma 8.2], hence L is pseudo-$\mathrm{AlgExt}(L,\mathcal{F}_L)$-closed. Thus, L is $P\mathcal{F}_LC$ (Data 6.1). Let $\beta\colon \mathrm{AlgExt}(L,\mathcal{F}_L)_{\min} \to \mathrm{Val}(L)$ be the map that maps the unique P-adic valuation w_F of each $F \in \mathrm{AlgExt}(L,\mathcal{F}_L)_{\min}$ onto $w_F|_L$. By the case $L = K$ (applied to $L,\mathcal{F}_L$ replacing $K,\mathcal{F}$), β is étale continuous. Each $F \in \mathrm{AlgExt}(L)\cap\mathrm{AlgExt}(K,\mathcal{F})_{\min}$ belongs to $\mathrm{AlgExt}(L,\mathcal{F}_L)_{\min}$. Moreover, the restriction of β to $\mathrm{AlgExt}(L)\cap\mathrm{AlgExt}(K,\mathbb{F})_{\min}$ coincides with λ_L. Consequently, λ_L is étale continuous. ∎

7. The Block Approximation Theorem for P-adic Valuations

We attach a field-valuation structure $\mathbf{K}_F$ to each P$\mathcal{F}$C field K. Then we reduce the P-adic Block Approximation Theorem to the Residue Bounded Block Approximation Theorem 4.1.

Construction 7.1: P-*adic Structure.* Let $\mathcal{F}$ be a finite set of P-adic fields closed under Galois isomorphism. Let K be a P$\mathcal{F}$C field. We attach a proper field-valuation structure $\mathbf{K}_{\mathcal{F}}$ to $\mathcal{F}$ and K.

Let $\mathcal{X} = \mathrm{AlgExt}(K, \mathcal{F})_{\min}$. By Lemma 6.2, $\mathcal{X}$ is étale profinite. Moreover, the action of $\mathrm{Gal}(K)$ on $\mathcal{X}$ by conjugation is étale continuous. We choose a homeomorphic copy X of $\mathcal{X}$ and a homeomorphism $\delta\colon X \to \mathcal{X}$. For each $x \in X$ let $K_x = \delta(x)$. We define a continuous action of $\mathrm{Gal}(K)$ on X via δ; that is, $K_{x^\sigma} = K_x^\sigma$ for all $\sigma \in \mathrm{Gal}(K)$. We denote the unique P-adic valuation of K_x [HJP05, Prop. 8.2(c)] by v_x. Then $v_x^\sigma = v_{x^\sigma}$ for all $x \in X$ and $\sigma \in \mathrm{Gal}(K)$. By Proposition 2.3(b), $\mathrm{Aut}(K_x/K) = 1$, so $\mathrm{Gal}(K_x) = \{\sigma \in \mathrm{Gal}(K) \mid x^\sigma = x\}$ for each $x \in X$.

Let L be a finite extension of K and set $X_L = \{x \in X \mid L \subseteq K_x\}$. Then $\delta(X_L) = \mathrm{AlgExt}(L) \cap \mathcal{X}$. By Lemma 6.3(c), the map $\lambda_L\colon \mathrm{AlgExt}(L) \cap \mathcal{X} \to \mathrm{Val}(L)$ is étale continuous. Hence, the map $\lambda_L \circ \delta\colon X_L \to \mathrm{Val}(L)$ mapping $x \in X_L$ to $v_x|_L$ is continuous.

It follows that $\mathbf{K}_{\mathcal{F}} = (K, X, K_x, v_x)_{x \in X}$ is a proper field-valuation structure (Section 4). ∎

THEOREM 7.2 (P-adic Block Approximation Theorem): *Let $\mathcal{F}$ be a finite set of* P-*adic fields closed under Galois isomorphism. Let K be a P$\mathcal{F}$C field. Then the field-valuation structure $\mathbf{K}_{\mathcal{F}}$ has the block approximation property.*

Proof: We use the notation of Construction 7.1. By assumption, K is P$\mathcal{X}$C. Let m a common multiple of the orders of the multiplicative groups of the residue fields of the fields in $\mathcal{F}$. Then (K_x, v_x) is m-bounded in the sense of Section 3 for each $x \in X$.

CLAIM: *For each $x \in X$ the valued field (K_x, v_x) is the Henselian closure of $(K, v_x|_K)$.* Indeed, as a P-adically closed field, (K_v, v_x) is Henselian [HJP05, Prop. 8.2(g)]. Hence, (K_x, v_x) is an extension of a Henselian closure (E, w) of $(K_x, v_x|_K)$. In particular,

$E \neq \tilde{K}$. Hence, by Proposition 5.6, $E \in \mathcal{X}$. The minimality of K_x implies that $K_x = E$. Thus, $v_x = w$ and (K_x, v_x) is the Henselian closure of $(K, v_x|_K)$, as claimed.

It follows from Theorem 4.1 that $\mathbf{K}_{\mathcal{F}}$ has the block approximation property. ∎

Finally we show how the version of the P-adic Block Approximation Theorem appearing in the introduction follows from Theorem 7.2.

Proof of the P*-adic Block Approximation Theorem of the Introduction:* Let $\mathbf{K}_{\mathcal{F}}$ and $\delta\colon X \to \mathcal{X}$ be as in Construction 7.1. For each $i \in I_0$ let $X_i = \delta^{-1}(\mathcal{X}_i)$. Then $(V, X_i, L_i, \mathbf{a}_i, c_i)_{i \in I_0}$ is a block approximation problem for $\mathbf{K}_{\mathcal{F}}$. By Theorem 7.2, this problem has a solution $\mathbf{a}$. It satisfies, $v_F(\mathbf{a} - \mathbf{a}_i) > v_F(c_i)$ for each $i \in I_0$ and every $F \in \mathcal{X}_i$. ∎

References

[Ax71] J. Ax, *A metamathematical approach to some problems in number theory,* AMS Proc. Symp. Pure Math. **XX** (1971), 161–190.

[Eng78] A. J. Engler, *The relative separable closure of a valued field in its completion,* manuscripta mathematica **24** (1978), 83–95.

[FHV94] M. D. Fried, D. Haran, H. Völklein, *Real hilbertianity and the field of totally real numbers,* Contemporary Mathematics **174** (1994), 1–34.

[FrJ86] M. D. Fried and M. Jarden, *Field Arithmetic,* Ergebnisse der Mathematik (3) **11**, Springer-Verlag, Heidelberg, 1986.

[FrJ08] M. D. Fried and M. Jarden, *Field Arithmetic, Third Edition, revised by Moshe Jarden,* Ergebnisse der Mathematik (3) **11**, Springer, Heidelberg, 2008.

[Gey94] W.-D. Geyer, *On the irreducibility of sums of rational functions with separated variables,* Israel Journal of Mathematics **85** (1994), 135–168.

[HaJ85] D. Haran and M. Jarden, *The absolute Galois group of a pseudo real closed field,* Annali della Scuola Normale Superiore — Pisa, Serie IV, **12** (1985), 449–489.

[HaJ88] D. Haran and M. Jarden, *The absolute Galois group of a pseudo p-adically closed field,* Journal für die reine und angewandte Mathematik **383** (1988), 147–206.

[HJP05] D. Haran, M. Jarden, and F. Pop, P-*adically projective groups as absolute Galois groups,* International Mathematics Research Notices **32** (2005), 1957-1995.

[HJP07] D. Haran, M. Jarden, and F. Pop, *Projective group structures as absolute Galois structures with block approximation,* Memoirs of AMS **189** (2007), 1–56,

[Jar91a] M. Jarden, *Algebraic realization of p-adically projective groups,* Compositio Mathematica **79** (1991), 21–62.

[Jar91b] M. Jarden, *Intersection of local algebraic extensions of a Hilbertian field (A. Barlotti et al., eds),* NATO ASI Series C **333**, 343–405, Kluwer, Dordrecht, 1991.

[JaR80] M. Jarden and P. Roquette, *The Nullstellensatz over p-adically closed fields,* Journal of the Mathematical Society of Japan **32** (1980), 425–460.

[Pop88] F. Pop, *Galoissche Kennzeichnung p-adisch abgeschlossener Körper,* Journal für die reine und angewandte Mathematik **392** (1988), 145–175.

[Pop90] F. Pop, *Classically projective groups,* preprint, Heidelberg, 1990.

[Pop94] F. Pop, *On Grothendieck's conjecture of birational anabelian geometry,* Annals of Mathematics **138** (1994), 145–182.

[Rbn64] P. Ribenboim, *Théorie des valuations,* Les Presses de l'Université de Montréal, Montréal, 1964.

13 September, 2010