Math 624 (Algebraic Geometry) / Problem Set 2

Algebraic sets (continued)

1) Prove or disprove:
   a) $U = K\setminus\{0\}$ is connected as a subset of $K^1$. Is the same true if we replace “connected” by “irreducible”?
   b) The same questions about $U = K^n\setminus V$, with $V$ some affine $k$-algebraic subset of $K^n$.
   - The same questions in the case $\dim(V) < n - 1$.

2) Let $V$ be an affine $k$-algebraic set. Prove/disprove the following:
   - $V$ is connected as a subset of $K^1$.
   - The same question about $U = K^n\setminus V$, with $V$ some affine $k$-algebraic subset of $K^n$.

3) Let $V \subseteq K^n$, $W \subseteq K^m$ be affine $k$-algebraic sets, and $k[V] = k[x_1, \ldots, x_n]$, $k[W] = k[y_1, \ldots, y_m]$ be their rings of global functions, respectively. Give a complete proof of the following assertion form the class:
   - Every $k$-morphisms $\phi : k[W] \to k[V]$ defines a morphism of algebraic sets $\varphi : V \to W$ by $\varphi := (f_1, \ldots, f_m)$, where $f_i : V \to K$ is defined by $f_i := \phi(y_i)$, $i = 1, \ldots, m$.

4) For each of the following maps describe the image and check whether the image is a (quasi) affine $k$-algebraic (sub)set of the codomain. And check whether the map in discussion is bijective, respectively a $k$-isomorphism, onto its image:
   - $K^\times \to K^2$, $a \mapsto (a, 1/a)$.
   - $K^1 \to K^2$, $a \mapsto (a^2, a^3)$.
   - Is the same true if we replace “connected” by “irreducible”?

The ring of rational functions on $V$

Given an affine $k$-algebraic subset $V$, the total ring of fractions of $k[V]$ is denoted $k(V)$, and is called the rings of rational functions on $V$. We denote by $k_V := k(V) \cap K$ the relative algebraic closure of $k$ in $k(V)$, and call it the field of constants of $k(V)$.

5) Let $V$ be a non-empty affine $k$-algebraic set. Prove/disprove the following:
   - $V$ is irreducible iff $k(V)$ is a field.
   - Let $V_1, \ldots, V_r$ be the irreducible components of $V$. Then $k(V) \cong k(V_1) \times \ldots \times k(V_r)$ as $k$-algebras.
   - $V$ is geometrically integral iff $k_V = k$. Respectively, $V$ is geometrically irreducible iff $k_V | k$ is purely inseparable.
   - In the context of Problem 6, how does relate connectivity to $k_V \subset k(V)$?

(Pre)sheaves

6) Let $\mathcal{P}$ be a presheaf on $X$ with values in one of the categories $C$ we considered.
   - Check the details of the fact that the associated sheaf $\iota : \mathcal{P} \to \widehat{\mathcal{P}}$ is indeed a sheaf.
   - Check that $\iota : \mathcal{P} \to \widehat{\mathcal{P}}$ has the universal property given in the class.

7) In the context/notations from Problem 5 above, let $U \subseteq X$ an open (non)empty set, and consider the restriction $\iota|U : \mathcal{P}|_U \to \widehat{\mathcal{P}}|_U$ of $\iota$ to $U$. Prove/disprove that the following are equivalent:
   - $\mathcal{P}|_U$ is a sheaf.
   - $\iota|U$ is an isomorphism.

8) Complete the proofs of assertions 1), 2), 3), 4) from the Gluing Lemma.