Due: March 2011

Math 625 (Algebraic Geometry) / Problem Set 5

In the next problems $k$ is a perfect field and all the curves are complete normal (thus projective and smooth) geometrically integral $k$-curves. Further, if $X$ is such a curve, then $g_X$ denotes the genus of $X$. Further, $\varphi : X \to Y$ is a finite non-constant morphism of such $k$-curves, and $\deg(\varphi)$ is the degree of $\varphi$.

1) Suppose that $g_Y = 5$ and $\deg(\varphi) = 3$.
   a) What are the possible values of $g_X$?
   b) Does the answer depend on $\text{char}(k)$?
   c) The same question, provided $\varphi$ is generically Galois.

2) Let $X$ be a hyperelliptic curve defined by completion of $X_0 := V(X_1^2 - p(X_1)) \subset \mathbb{A}^2_k$, where $p(X_1)$ is a separable polynomial of degree $n$. Prove/disprove:
   a) The completion of $X_0$ in $\mathbb{P}^2_k$ under $\mathbb{A}^2_k \hookrightarrow \mathbb{P}^2_k$ is smooth iff $n = 2m + 1$ is odd.
   b) Let $X$ and $Y$ be hyperelliptic curves defined by $X_1^2 - p(X_1), X_2^2 - q(X_1) \in k[X_1, X_2]$, respectively, with $p$ and $q$ separable and of degree $n = 2m + 1$. Then $X \cong Y$ as $k$-curves iff $q(X_1) = c^2 p(X_1)$ for some $c \in k$ a square.

3) Let $X$ be a hyperelliptic curve (thus $g_X > 1$).
   a) Prove the assertion from the class, that the rational function subfield $k(t) \subset k(X)$ with $[k(X) : k(t)] = 2$ is unique.
   b) Deduce from the above that $\text{Aut}(X)$ is finite.
   c) Can you give a precise description of $\text{Aut}(X)$?

4) Let $\text{char} = p > 0$.
   a) Show that there exist finite unramified covers $X \to \mathbb{A}^1_k$ of arbitrary large genus.
   b) Is the same the case if $\text{char}(k) = 0$?

[Hint: If $\mathbb{A}^1_k = \text{Spec} k[x]$, then $y^p - y = x^m$ define unramified covers of $\mathbb{A}^1_k$ (WHY?), etc.]

5) Let $\varphi : X \to Y$, $P \mapsto Q$, be a finite generically Galois cover with automorphism group $G$. Denote by $S \subset X$ and $T \subset Y$ the ramification, respectively branch, loci of $\varphi$. Suppose that $\text{char}(k) = 0$. Using the Hurwitz genus formula, prove the assertions from the class:
   a) $2g_X - 2 = |G| [2g_Y - 2 + \sum_{Q \in T} (1 - 1/e_Q)]$.
   b) Let $g_X > 1$. Deduce from the above that $|G|$ is maximized in the case $g_Y = 0$ and $|T| = 3$ and the three ramification indices $e_Q$, $Q \in T$, are $1/2, 1/3, 1/7$.

- Problems 2.2, 2.3, 2.4 from Ch. IV of Hartshorne’s book *Algebraic Geometry*. (Try to do as much as possible over a non-algebraically closed base field $k$.)