Due: Friday, Nov 17, 2017 (at noon)

Math 202 / Midterm 2 (two pages)

Academic Integrity Statement:

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this Math 202 exam. [That means, among other things, that you are allowed to: (a) discuss the problems of the midterm with your colleagues, but not work out solutions together; (b) ask any member the Math Dept about hints to the exam problems, but you must *first* mention to her/him that it goes about problems on a take home exam; (c) consult resources (books, Internet, etc.), but you must design/write down your own proofs.]

Name (printed): ________________________ Signature:

Note: There are 7 (seven) problems on this exam.
Points: 14.5 points for each problem (extra and/or partial credit possible).
Grading: 55 < C−, C, C+ < 70 < B−, B, B+ < 85 < A−, A, A+
Procedures: Print out the two pages of the exam and staple them to your work. Write your name (printed) and sign the above Academic Integrity Statement.

• Recall: A complete proof must contain all the necessary explanations/steps, and in order to *disprove* an assertion you must give a counterexample showing that the assertion is not true.

1) Consider the assertion in plain English:

($*$) *Every positive rational number less than $\frac{1}{11}$ is a sum of four squares of rational numbers.*

a) Write the above assertion using quantifiers.
b) What is the negation of the above assertion in plain English.
c) Write the negation of the above assertion using quantifiers.
d) Is the above assertion true?

2) Let $Z \subset \mathbb{Z}$ and $Q \subset \mathbb{Q}$ be subsets. Prove/disprove the following:

a) Suppose that $Z \neq \{0_{\mathbb{Z}}\}$, and that every equation of the form $x + a = b$ with $a, b \in Z$ has some solution $x \in Z$. Then $Z = \mathbb{Z}$.
b) Suppose that $Q \neq \{0_{\mathbb{Q}}\}$, and that every equation of the form $cx + a = b$ with $a, b, c \in Q$, $c \neq 0_{\mathbb{Q}}$ has some solution $x \in Q$. Then $Q = \mathbb{Q}$.

3) For a commutative ring $R, +, \cdot$ and $a, b, c, d \in R$, using $+$ and $\cdot$ define on R a new "addition" by $x \boldsymbol{+} y = x + y + a$ and a new "multiplication" by $x \bullet y = xy + bx + cy + d$.

a) In each of the cases $R = \mathbb{Z}$ or $R = \mathbb{R}$, find all $a, b, c, d \in R$ such that $R, \boldsymbol{+}, \bullet$ is a ring.
b) Solve in the ring $R, \boldsymbol{+}, \bullet$ the equations $x^2 \boldsymbol{+} 3 \bullet x = 1_R$ and $x^2 \boldsymbol{+} 3 \bullet x = 0_R$.

4) Write the negations of the following assertions using quatifiers:

a) Every bounded sequence $(x_n)_n$ is convergent.

b) Every bounded sequence $(x_n)_n$ is Cauchy.

c) If the sum of two sequences $(x_n)_n$, $(y_n)_n$ is convergent, then $(x_n)_n$ and $(y_n)_n$ are convergent.

5) Let $(x_n)_n$, $(y_n)_n$, $(z_n)_n$ be sequences of rational numbers. Answer/prove/disprove:

a) If $(x_n)_n$, $(x_n)_n + (y_n)_n$, $(x_n)_n + (y_n)_n + (z_n)_n$ are Cauchy, then $(z_n)_n$ is Cauchy.

b) If $(x_n)_n$, $(x_n)_n + (y_n)_n$, $(x_n)_n + (y_n)_n + (z_n)_n$ are bounded, then $(z_n)_n$ is bounded.

c) If $x_n \to 0$ and $(y_n)_n$ is bounded, then $(x_n)_n \cdot (y_n)_n$ is convergent.

d) If $(x_n)_n$ is convergent and $(y_n)_n$ is bounded, then $(x_n)_n \cdot (y_n)_n$ is convergent.

6) Let $(X_n)_n$ be a sequence of subsets $X_n \subset \mathbb{Q}$ satisfying the three conditions: (i) $X_{n+1} \subset X_n$ for all $n \in \mathbb{N}$; (ii) $X := \cap_n X_n$ is non-empty; (iii) X_0 bounded below. Prove the following:

a) $x := \inf(X)$ and $x_n := \inf(X_n)$ exist in $\mathbb{R}$ for all $n \in \mathbb{N}$.

b) $(x_n)_n$ is an increasing sequence, and $x_n \to x$.

[**Hint:** First, all the sets X_n are bounded below (WHY), etc … Second, if $Z \subset Y \subset \mathbb{R}$ are non-empty subsets, and $\inf(Y)$ exists, then $\inf(Z)$ exists (WHY), and $\inf(Z) \geqslant \inf(Y)$ (WHY), etc …]

7) Answer/prove/disprove the following:

a) Let $p(t) \in \mathbb{R}[t]$ be a fixed polynomial. Then $x_n := p(n)/n^n$ is convergent to 0.

b) For $x_0 \in \mathbb{R}$, set $x_{n+1} := x_n^2 - 2$ for $n \in \mathbb{N}$. For which values of $x_0 \in \mathbb{R}$ one has that $(x_n)_n$ is bounded, respectively convergent?

c) Let $(x_n)_n$ and $(y_n)_n$ be sequences of real numbers satisfying $n\, y_n = x_1 + \cdots + x_n$ for all $n \in \mathbb{N}_{>0}$. Then $(x_n)_n$ is bounded, respectively convergent if and only if $(y_n)_n$ is so.

[**Hints:** To a): For any fixed $k \in \mathbb{N}$ one has: $n^k/n^n \to 0$ as $n \to \infty$ (WHY), etc … To b): How does compare x to $f(x) = x^2 - 2$, etc … To c): If $(x_n)_n$ is bounded/convergent, so is $(y_n)_n$ (WHY). For the converse, find the right (counter?)examples …]