**Math 202 / Problem Set 10 (two pages)**

**Due: Mo, Nov 27, 2017**

Recall: (Google it!) Definition of a topology and a topological space \( X, \tau_X \); distance map and metric space \( X, d_X \) and the topology \( \tau_{d_X} \) defined by the distance function \( d_X \) on \( X \); hence \( U \in \tau_{d_X} \iff U \) is an (arbitrary) union open balls \( B(x_i, \epsilon_i), i \in I \). Further, recall that the \( U \in \tau_X \) are called open sets, and their complements \( T = U \setminus U \subset X \) are called closed sets.

1) In the above notations, prove/disprove/answer the following:
   a) If \( U_1, \ldots, U_n \in X \) are neighborhoods of \( x \in X \), then \( \cap_{i=1}^n U_i \) is neighborhood of \( x \).
   b) A subset \( U \subset X \) is open iff \( U \) is neighborhood for all \( x \in U \).

2) Let \( X, d_X \) be a metric space. Prove the following assertions from the class:
   a) If \( x' \in B_X(x, \epsilon) \) and \( x' := \epsilon - d(x, x') \), then \( \epsilon' > 0 \), and \( B(x', \epsilon') \subset B(x, \epsilon) \).
   b) \( U \subset X \) is open iff for every \( x \in X \) there exists \( \epsilon_x > 0 \) such that \( B(x, \epsilon_x) \subset U \).
   c) A finite intersection of open balls is an open subset.

Recall that the \( xy \)-coordinate plane \( X := \mathbb{R}^2 \) is a metric space w.r.t. the Euclidean metric defined by \( d_E(P_1, P_2) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \) for all points \( P_1 := (x_1, y_1), P_2 := (x_2, y_2) \) of \( X \). In these notation, define the following two maps \( d_1, d_\infty : X \times X \to \mathbb{R} \) by
   \[
   d_1(P_1, P_2) := |x_1 - y_1| + |x_2 - y_2|, \quad d_\infty(P_1, P_2) := \max(|x_1 - y_1|, |x_2 - y_2|).
   \]

3) Prove the following:
   a) The maps \( d_1, d_\infty \) are distance maps on \( X \).
   b) For all \( P_1, P_2 \) one has: \( d_\infty(P_1, P_2) \leq d_E(P_1, P_2) \leq d_1(P_1, P_2) \leq 2d_\infty(P_1, P_2) \).
   c) Draw the open balls \( B_{d_1}(O, 1), B_{d_E}(O, 1), B_{d_\infty}(O, 1) \), where \( O \) is the origin.
   d) Prove that the topologies \( \tau_{d_1}, \tau_{d_E} \) and \( \tau_{d_\infty} \) are the same.

**[Hint to d): By Problem 2), b), a subset \( U \subset X = \mathbb{R}^2 \) is open iff for every \( P \in U \) there exists an open ball in \( U \) centered at \( P \); and by the inequalities from b) above, one has: \( B_{d_\infty}(P, \epsilon) \subset B_{d_1}(P, \epsilon) \subset B_{d_E}(P, \epsilon) \subset B_{d_\infty}(P, \epsilon) \) [WHY], etc…]**

**Therefore**, there can be several different distance maps defining the same topology!

4) Let \( f : X \to Y \) be a map of topological spaces. Prove in all detail the assertion from the class that the following are equivalent:
   i) \( f \) is continuous.
   ii) For every \( V \subset Y \) open in \( Y \), the preimage \( U := f^{-1}(V) \subset X \) is open in \( X \).
   iii) For every \( T \subset Y \) closed in \( Y \), the preimage \( S := f^{-1}(T) \subset X \) is closed in \( X \).

**[Hint: By definition, \( V \subset Y \) is open \( \iff T := Y \setminus V \) is closed, and vice-versa; and \( f^{-1}(Y \setminus V) = X \setminus f^{-1}(V) \) [WHY], etc…]**

**The ring of continuous maps:** Let \( X \) be an arbitrary set, and recall that the set of all the maps \( \text{Maps}(X, \mathbb{R}) \) endowed with the usual addition of maps \( f \oplus g \) and multiplication of maps \( f \bullet g \) is a commutative ring, having the constant zero map \( 0_{\text{Maps}} \) as neutral element for \( \oplus \), and the constant one map \( 1_{\text{Maps}} \) as neutral element for multiplication.

5) Suppose that \( X \) is a topological space. Prove in all detail the following assertions:
   a) If \( f, g : X \to \mathbb{R} \) are continuous at \( x \in X \), then \( f \oplus g \) and \( f \bullet g \) are continuous at \( x \).
b) If \( f \) is continuous at \( x \in X \), and \( f(x) \neq 0 \), then \( x \) has a neighborhood \( U_x \subset X \) such that \( f(x') \neq 0 \) for \( x' \in U_x \) and \( \frac{1}{f} : U_x \to \mathbb{R} \) is continuous at \( x \in U \subset X \).

c) The set \( C(X, \mathbb{R}) \subset \text{Maps}(X, \mathbb{R}) \) of all the continuous maps is a ring w.r.t. the addition \( \oplus \) and the multiplication \( \cdot \). Further, if \( f \in C(X, \mathbb{R}) \) satisfies \( f(x) \neq 0 \) for all \( x \in X \), then \( \frac{1}{f} : X \to \mathbb{R} \) is continuous on \( X \).

**Intervals:** Recall that the completed real line is \( \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\} \), and one extends the ordering \( \leq \) of \( \mathbb{R} \) to \( \overline{\mathbb{R}} \) by setting: \( -\infty < x < \infty \) for all \( x \in \mathbb{R} \). Open intervals of \( \mathbb{R} \) are the subsets \( (a, b) := \{ x \in \mathbb{R} \mid a < x < b \} \), where \( a, b \in \mathbb{R} \). Closed intervals of \( \mathbb{R} \) are the subsets \( [a, b] := \{ x \in \mathbb{R} \mid a \leq x \leq b \} \), for \( a, b \in \mathbb{R} \).

6) Let \( \mathbb{R}[t] \) be the ring of polynomials with real coefficients, and for \( p(t) \in \mathbb{R}[t] \) recall the polynomial function \( f_p : \mathbb{R} \to \mathbb{R} \) by \( f_p(x) = p(x) \) for \( x \in \mathbb{R} \). Prove/answer the following:

a) \( f_p \) is continuous for every \( p(t) \in \mathbb{R}[t] \), and \( f_p(\mathbb{R}) \) is an unbounded interval in \( \mathbb{R} \).

b) Give the behavior of \( f_p(x) \) as \( x \to -\infty \), respectively \( x \to \infty \), and describe \( f_p(\mathbb{R}) \).

7) For \( n \in \mathbb{N}_{\geq 0} \), consider the function \( f_n : \mathbb{R} \to \mathbb{R} \) defined by \( f_n(x) := x^n \) called the power \( n \) map. Then \( f_n \) is a polynomial map \([WHY]\), hence continuous. Prove/answer the following:

a) Let \( n = 2m + 1 \). Then \( f_n : \mathbb{R} \to \mathbb{R} \) is strictly increasing, and \( f_n(\mathbb{R}) = \mathbb{R} \).

In particular, \( \forall \ y \in \mathbb{R} \exists \text{ unique } x \in \mathbb{R} \text{ s.t. } x^n = y \). Notation. \( x := \sqrt[2m+1]{y} \).

b) Let \( n = 2m \). Then \( f_n : [0, \infty) \to [0, \infty) \) is strictly increasing, \( f_n([0, \infty)) = [0, \infty) \).

In particular, \( \forall \ y \in [0, \infty) \exists \text{ unique } x \in [0, \infty) \text{ s.t. } x^n = y \). Notation. \( x := \sqrt[2m]{y} \).

c) The \( n^{th} \) root map \( \sqrt[n]{\cdot} : [0, \infty) \to [0, \infty) \) by \( x \mapsto \sqrt[n]{x} \) is continuous and strictly increasing.

[Hint. To a): \( x^{2m} \geq 0 \), hence \( x < y \) implies: \( x^{2m} < y^{2m} \), hence \( x^{2m+1} = x x^{2m} < y y^{2m} \). One has \( \mathbb{R} = \cup_{k \in \mathbb{N}} (-k, k) \), and \( (-k)^{2m+1} < -k < k^{2m+1} \), etc...Finally, us the Intermediate Value Thm, etc...To b): Argue similarly...]

**Remark.** \( f_n \) and \( \sqrt[n]{\cdot} \) are inverse to each other w.r.t. composition of function \([WHY]\).

The power function and the exponential function

By the above discussion, for every non-negative real number \( y_0 \in \mathbb{R}_{\geq 0} \), there exists a unique \( x \in \mathbb{R}_{\geq 0} \) such that \( x^n = y_0 \), namely, \( x := \sqrt[n]{y_0} \). In particular, given \( m \in \mathbb{Z} \), the number \( y_1 := \sqrt[m]{y_0} \) is a well defined real number \([WHY]\). Notations. \( y_0 = x^n \) and \( y_1 := \sqrt[m]{y_0} \).

8) For \( u, v > 0 \) real numbers, \( \frac{m'}{n'} \) and \( \frac{m}{n} \) rational numbers as above, prove:

a) \( \sqrt[n]{u^{m'}} = (u^{1/n})^{m'} = (u^{m/n})^{m'}; \quad (uv)^{m/n} = u^{m/n} v^{m/n}; \quad u^{m/n} u^{m'/n'} = u^{m/n + m'/n'}; \quad (u^{m/n})^{m'/n'} = u^{m/n m'/n'} \)

b) Let \( (x_n)_n, x_n \in \mathbb{Q} \), satisfy \( x_n \to x \in \mathbb{R} \). Then \( (u^{x_n})_n \) is Cauchy, and \( u^x := \lim_n u^{x_n} \) depends on \( x = \lim_n x_n \) only.

9) For \( \alpha \in \mathbb{R} \), the power-\( \alpha \) function \( f_\alpha : (0, \infty) \to (0, \infty) \), \( f_\alpha(x) := x^\alpha \) is continuous and:

a) \( f_0(x) = x^0 = 1 \) for all \( x \in (0, \infty) \).

b) \( f_\alpha \) is strictly increasing if \( \alpha > 0 \), respectively strictly decreasing if \( \alpha < 0 \).

10) The exponential function \( \exp_a : \mathbb{R} \to (0, \infty) \), \( \exp_a(x) := a^x \) in basis \( a > 0 \) is continuous, and:

a) \( \exp_1(x) = 1^x = 1 \) for all \( x \in \mathbb{R} \).

b) \( \exp_a \) is strictly increasing if \( a > 1 \), respectively strictly decreasing if \( 0 < a < 1 \).