Due: Mo, Nov 27, 2017

Math 202 / Problem Set 10 (two pages)

Recall: (Google it!) Definition of a *topology* and a *topological space* X, τ_X; *distance map* and *metric space* X, d_X and the topology τ_{d_X} defined by the distance function d_X on X; hence $U \in \tau_{d_X} \overset{\text{def}}{\longleftrightarrow} U$ is an (arbitrary) union open balls $B(x_i, \epsilon_i)$, $i \in I$. Further, recall that the $U \in \tau_X$ are called *open sets*, and their complements $T = U \backslash U \subset X$ are called *closed sets*.

1) In the above notations, prove/disprove/answer the following:

a) If $U_1, \ldots, U_n \in X$ are neighborhoods of $x \in X$, then $\cap_{i=1}^n U_i$ is neighborhood of x.
b) A subset $U \subset X$ is open iff U is neighborhood for all $x \in U$.

2) Let X, d_X be a metric space. Prove the following assertions from the class:

a) If $x' \in B_X(x, \epsilon)$ and $\epsilon' := \epsilon - d(x, x')$, then $\epsilon' > 0$, and $B(x', \epsilon') \subset B(x, \epsilon)$.
b) $U \subset X$ is open iff for every $x \in X$ there exists $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset U$.
c) A finite intersection of open balls is an open subset.

Recall that the xy-coordinate plane $X := \mathbb{R}^2$ is a metric space w.r.t. the *Euclidean metric* define by $d_E(P_1, P_2) :\overset{\text{def}}{=} \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ for all points $P_1 := (x_1, y_1)$, $P_2 := (x_2, y_2)$ of X. In these notation, define the following two maps $d_1, d_\infty : X \times X \to \mathbb{R}$ by

$$d_1(P_1, P_2) :\overset{\text{def}}{=} |x_1 - y_1| + |x_2 - y_2|, \quad d_\infty(P_1, P_2) :\overset{\text{def}}{=} \max(|x_1 - y_1|, |x_2 - y_2|).$$

3) Prove/answer the following:

a) The maps d_1, d_2 are distance maps on X.
b) For all P_1, P_2 one has: $d_\infty(P_1, P_2) \leqslant d_E(P_1, P_2) \leqslant d_1(P_1, P_2) \leqslant 2\, d_\infty(P_1, P_2)$.
c) Draw the open balls $B_{d_1}(O, 1)$, $B_{d_E}(O, 1)$, $B_{d_\infty}(O, 1)$, where O is the origin.
d) Prove that the topologies τ_{d_1}, τ_{d_E} and τ_{d_∞} are the same.

[**Hint** to d): By Problem 2), b), a subset $U \subset X = \mathbb{R}^2$ is open iff for every $P \in U$ there exists an open ball in U centered at P; and by the inequalities from b) above, one has: $B_{d_\infty}(P, \frac{1}{2}\epsilon) \subset B_{d_1}(P, \epsilon) \subset B_{d_E}(P, \epsilon) \subset B_{d_\infty}(P, \epsilon)$ (WHY), etc...]

Therefore, there can be several different distance maps defining the same topology!

4) Let $f : X \to Y$ be a map of topological spaces. Prove in all detail the assertion from the class the fact that the following are equivalent:

i) f is continuous.
ii) For every $V \subset Y$ open in Y, the preimage $U := f^{-1}(V) \subset X$ is open in X.
iii) For every $T \subset Y$ closed in Y, the preimage $S := f^{-1}(T) \subset X$ is closed in X.

[**Hint**: By definition, $V \subset Y$ is open $\overset{\text{def}}{\longleftrightarrow}$ $T := Y \backslash V$ is closed, and vice-versa; and $f^{-1}(Y \backslash Y') = X \backslash f^{-1}(Y')$ (WHY), etc...]

The ring of continuous maps: Let X be an arbitrary set, and recall that the set of all the maps $\mathrm{Maps}(X, \mathbb{R})$ endowed with the usual addition of maps $f + g$ and multiplication of maps $f \bullet g$ is a commutative ring, having the constant zero map 0_{Maps} as neutral element for $+$, and the constant one map 1_{Maps} as neutral element for multiplication.

5) Suppose that X is a topological space. Prove in all detail the following assertions:

a) If $f, g : X \to \mathbb{R}$ are continuous at $x \in X$, then $f + g$ and $f \bullet g$ are continuous at x.

b) If f is continuous at $x \in X$, and $f(x) \neq 0$, then x has a neighborhood $U_x \subset X$ such that $f(x') \neq 0$ for $x' \in U_x$ and $\frac{1}{f} : U_x \to \mathbb{R}$ is continuous at $x \in U \subset X$.

c) The set $\mathcal{C}(X, \mathbb{R}) \subset \mathrm{Maps}(X, \mathbb{R})$ of all the continuous maps is a ring w.r.t. the addition $+$ and the multiplication $\bullet$. Further, if $f \in \mathcal{C}(X, \mathbb{R})$ satisfies $f(x) \neq 0$ for all $x \in X$, then $\frac{1}{f} : X \to \mathbb{R}$ is continuous on X.

Intervals: Recall that the *completed real line* is $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, and one extends the ordering $\leqslant$ of $\mathbb{R}$ to $\overline{\mathbb{R}}$ by setting: $-\infty < x < \infty$ for all $x \in \mathbb{R}$. *Open intervals* of $\mathbb{R}$ are the subsets $(a, b) :\overset{\text{def}}{=} \{x \in \mathbb{R} \mid a < x < b\}$, where $a, b \in \overline{\mathbb{R}}$. *Closed intervals* of $\mathbb{R}$ are the subsets $(-\infty, a] :\overset{\text{def}}{=} \{x \in \mathbb{R} \mid x \leqslant a\}$, $[a, b] :\overset{\text{def}}{=} \{x \mid a \leqslant x \leqslant b\}$, $[b, \infty) :\overset{\text{def}}{=} \{x \mid b \leqslant x\}$, for $a, b \in \mathbb{R}$.

6) Let $\mathbb{R}[t]$ be the ring of polynomials with real coefficients, and for $p(t) \in \mathbb{R}[t]$ recall the polynomial function $f_p : \mathbb{R} \to \mathbb{R}$ by $f_p(x) = p(x)$ for $x \in \mathbb{R}$. Prove/answer the following:

a) f_p is continuous for every $p(t) \in \mathbb{R}[t]$, and $f_p(\mathbb{R})$ is an unbounded interval in $\mathbb{R}$.

b) Give the behavior of $f_p(x)$ as $x \to -\infty$, respectively $x \to \infty$, and describe $f_p(\mathbb{R})$.

7) For $n \in \mathbb{N}_{>0}$, consider the function $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) := x^n$ called the power n map. Then f_n is a polynomial map (WHY), hence continuous. Prove/answer the following:

a) Let $n = 2m + 1$. Then $f_n : \mathbb{R} \to \mathbb{R}$ is strictly increasing, and $f_n(\mathbb{R}) = \mathbb{R}$.
In particular, $\forall\, y \in \mathbb{R}\ \exists$ **unique** $x \in \mathbb{R}$ s.t. $x^n = y$. **Notation**. $x :\overset{\text{def}}{=} \sqrt[n]{y}$.

b) Let $n = 2m$. Then $f_n : [0, \infty) \to [0, \infty)$ is strictly increasing, $f_n([0, \infty)) = [0, \infty)$.
In particular, $\forall\, y \in [0, \infty)\ \exists$ **unique** $x \in [0, \infty)$ s.t. $x^n = y$. **Notation**. $x :\overset{\text{def}}{=} \sqrt[n]{y}$.

c) The nth root map $\sqrt[n]{\ } : [0, \infty) \to [0, \infty)$ by $x \mapsto \sqrt[n]{x}$ is continuous and strictly increasing.

[**Hint**. To a): $x^{2m} \geq 0$, hence $x < y$ implies: $x^{2m} < y^{2m}$, hence $x^{2m+1} = x\, x^{2m} < y\, y^{2m}$ (WHY). One has $\mathbb{R} = \cup_{k \in \mathbb{N}}(-k, k)$, and $(-k)^{2m+1} < -k < k < k^{2m+1}$ (WHY), etc... Finally, us the Intermediate Value Thm, etc... To b): Argue similarly...]

Remark. f_n and $\sqrt[n]{\ }$ are *inverse to each other w.r.t. composition of function* (WHY).

The power function and the exponential function

By the above discussion, for every non-negative real number $x \in \mathbb{R}_{\geqslant 0}$, there exists a unique $y_0 \in \mathbb{R}_{\geqslant 0}$ such that $x = y_0^n$, namely, $y_0 :\overset{\text{def}}{=} \sqrt[n]{x}$. In particular, given $m \in \mathbb{Z}$, the number $y_1 :\overset{\text{def}}{=} y^m \in \mathbb{R}_{\geqslant 0}$ is a well defined real number (WHY). **Notations**. $y_0 = x^{\frac{1}{n}}$ and $y_1 :\overset{\text{def}}{=} x^{\frac{m}{n}}$.

8) For $u, v > 0$ real numbers, $\frac{m'}{n'}$ and $\frac{m}{n}$ rational numbers as above, prove:

a) $u^{\frac{m}{n}} = (u^m)^{\frac{1}{n}} = (u^{\frac{1}{n}})^m$; $(uv)^{\frac{m}{n}} = u^{\frac{m}{n}} v^{\frac{m}{n}}$; $u^{\frac{m}{n}} u^{\frac{m'}{n'}} = u^{\frac{m}{n} + \frac{m'}{n'}}$; $\left(u^{\frac{m}{n}}\right)^{\frac{m'}{n'}} = u^{\frac{m}{n}\frac{m'}{n'}}$

b) Let $(x_n)_n$, $x_n \in \mathbb{Q}$, satisfy $x_n \to x \in \mathbb{R}$. Then $(u^{x_n})_n$ is Cauchy, and $u^x :\overset{\text{def}}{=} \lim_n u^{x_n}$ depends on $x = \lim_n x_n$ only.

9) For $\alpha \in \mathbb{R}$, the *power-α function* $f_\alpha : (0,\infty) \to (0,\infty)$, $f_\alpha(x) :\overset{\text{def}}{=} x^\alpha$ is continuous and:

a) $f_0(x) = x^0 = 1$ for all $x \in (0, \infty)$.

b) f_α is strictly increasing if $\alpha > 0$, respectively strictly decreasing if $\alpha < 0$.

10) The *exponential function* $\exp_a : \mathbb{R} \to (0, \infty)$, $\exp_a(x) :\overset{\text{def}}{=} a^x$ *in basis* $a > 0$ is continuous, and

a) $\exp_1(x) = 1^x = 1$ for all $x \in \mathbb{R}$.

b) $\exp_a$ is strictly increasing if $a > 1$, respectively strictly decreasing if $0 < a < 1$.