Due: Tu, Sept 19, 2017 (in class)

Math 202 / Problem Set 2 (two pages)

Basics

1) Consider the assertions in plain English: $p \equiv$ (all doors will open), $q \equiv$ (the train stops). Answer the following:

a) What is the (logical) negation in plain English of the assertion p, respectively q.

b) Using $\neg$, $\&$, $\vee$ and p, q, write down the following assertion:
"Either all doors will open, or the train does not stop."

2) Let p, q be the assertions $p \equiv$ (more jobs), $q \equiv$ (lower taxes), $r \equiv$ (increase spending). In country X, statistical data show that lately more jobs were created. Explain what is logically faulty with the following assertions —dear to some politicians/economists/others:

a) "You see, since we lowered taxes, more jobs were cretated."

b) "You see, because we did not lower the taxes, we could increase spending, and therefore more jobs were created."

[**Hint:** Write down the assertion a), b) as logical assertions using p, q, r, and see whether the cause-effect is the one explained by politicians/economists/some; recall that "wrong implies everything"...]

3) Let A, B, C, D be given sets, and x be elements, e.g., real numbers. Answer the following:

a) Using $\cup$, $\cap$, $\setminus$ and A, B, C, D write down the sets of all x which satisfy:
i) $\Big(x \in A \text{ or } x \in B\Big)\ \&\ x \in C\ \&\ x \notin D$; ii) $x \in A$ or $\Big(x \in B\ \&\ x \in C\Big)\ \&\ x \notin D$.

b) Write as a union of disjoint intervals the sets of the real numbers $x \in \mathbb{R}$ satisfying:
i) $\Big(x < 20\ \&\ x^2 < 100\Big)$ or $x \notin (-\infty, -1]$; ii) $x < 20\ \&\ \Big(x^2 < 100 \text{ or } x \notin (-\infty, -1]\Big)$.

$(*)$ Does the place of the parentheses matter?

4) Let A, B be sets. Answer the following:

a) $\exists\, f : A \to B$ injective iff $\exists\, g : B \to A$ surjective.

b) $f : A \to B$ is injective iff $\exists\, g : B \to A$ surjective satisfying $g\Big(f(x)\Big) = x$ for all $x \in A$.

c) $f : A \to B$ is surjective iff $\exists\, g : B \to A$ injective satisfying $f\Big(g(y)\Big) = y$ for all $y \in B$.

Cardinality of sets. Recall that the cardinality of a set A, denoted by $|A|$, is, *intuitively,* a kind of size of A. Recall that $|A| \leq |B| \overset{\text{Def}}{\Longleftrightarrow} \exists\, f : A \to B$ injective, and that one has:

Theorem (Cantor, Bernstein, Schroeder)**.** *$|A| \leq |B|$ and $|B| \leq |A|$ if and only if there exists $f : A \to B$ bijective.*

Definition. Let X be a set. Recall the following:

- X is called finite of cardinality $|X| = n \geqslant 0$, if either $X = \emptyset$, and then $|X| := 0$, or $\exists\, f : \{1, \ldots, n\} \to X$ bijective, thus $X = \{x_1, \ldots, x_n\}$, where $x_i := f(i)$, $1 \leqslant i \leqslant n$.
- X is called countable, if $|X| = |\mathbb{N}|$, i.e., there exists a bijection $f : X \to \mathbb{N}$.
- X is called at most countable, if $|X| \leq |\mathbb{N}|$.

5) Let X be a non-empty set. Prove/disprove the following:

a) If X is finite, then every injective (resp. surjective) map $f : X \to X$ is bijective.

b) If X is infinite, there exists injective (surjective) $f : X \to X$ which are not bijective.

6) Let X be an arbitrary set, and $\mathcal{P}(X) := \{A \mid A \subseteq X\}$ be the power set of X. Prove:

a) If $|X| = n$ is finite, then $|\mathcal{P}(X)| = 2^n$.

b) One has always: $|X| < |\mathcal{P}(X)|$. Deduce from this that $|\mathbb{N}| < |\mathbb{R}|$.

[**Hint** to the second part of b): Define $f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ by $f(A) := \overline{a_0 \,.\, a_1 a_2 \ldots a_n \ldots}$ for $A \subset N$, where $a_n = 1$ if $n \in A$, and $a_n = 0$ if $n \notin A$. Then $A \neq A'$ implies $x_A \neq x_{A'}$ (WHY), hence f is injective, etc...]

7) Let X, Y be finite sets, say $|X| = m$ and $|Y| = n$. Prove/disprove the following assertions:

a) $|X \cup Y| + |X \cap Y| = |X| + |Y|$. What is the corresponding assertion for $|X \cup Y \cup Z|$?

b) $|X \times Y| = |X| \cdot |Y|$. What is the corresponding assertion for $|X \times Y \times Z|$?

8) Let A, B, A_n, $n \in \mathbb{N}$ be at most countable sets. Prove the following assertions:

a) $A \times B$ is at most countable. Is the same true for $A_0 \times \cdots \times A_n$ for all $n \in \mathbb{N}$.

b) $A \cup B$ is at most countable. Is the same true for $\cup_{n \in \mathbb{N}} A_n$?

c)* Is the same true for the (infinite cartesian) product $A_0 \times \cdots \times A_n \times \ldots$?

9) Complete the proof of the assertions from the class:

a) $+$ and $\cdot$ in $\mathbb{N}$ are associative, commutative, and 0, respectively 1 are neutral elements. Further, $\cdot$ is distributive w.r.t to $+$

b) $+$ and $\cdot$ have cancelation property in $\mathbb{N}$, respectively $\mathbb{N}_{>0}$, i.e., for $n, m, k \in \mathbb{N}$ one has:

- $m + k = n + k \Rightarrow m = n$.
- $m \cdot k = n \cdot k \Rightarrow m = n$, provided $k > 0$.

10) Recall that for $m, n \in \mathbb{N}$, one says that $m \leqslant n$ if there exists $k \in \mathbb{N}$ such that $n = m + k$. Complete the proof of the assertions from the class:

$$\leqslant \textit{ is compatible with } + \textit{ and } \cdot$$

That is, for all $n, m, k \in \mathbb{N}$ one has:

a) $m \leqslant n$ iff $m + k \leqslant n + k$.

b) $m \leqslant n$ iff $m \cdot k \leqslant n \cdot k$, provided $k > 0$.