Due: Tu, Sept 19, 2017 (in class)

Math 202 / Problem Set 2 (two pages)

Basics

1) Consider the assertions in plain English: \( p \equiv \) (all doors will open), \( q \equiv \) (the train stops). Answer the following:

a) What is the (logical) negation in plain English of the assertion \( p \), respectively \( q \).

b) Using \( \neg \), \( \& \), \( \lor \) and \( p \), \( q \), write down the following assertion:
   “Either all doors will open, or the train does not stop.”

2) Let \( p \), \( q \) be the assertions \( p \equiv \) (more jobs), \( q \equiv \) (lower taxes), \( r \equiv \) (increase spending).

In country \( X \), statistical data show that lately more jobs were created. Explain what is logically faulty with the following assertions — dear to some politicians/economists/others:

a) “You see, since we lowered taxes, more jobs were created."

b) “You see, because we did not lower the taxes, we could increase spending, and therefore more jobs were created.”

[Hint: Write down the assertion a), b) as logical assertions using \( p \), \( q \), \( r \), and see whether the cause-effect is the one explained by politicians/economists/some; recall that “wrong implies everything”…]

3) Let \( A \), \( B \), \( C \), \( D \) be given sets, and \( x \) be elements, e.g., real numbers. Answer the following:

a) Using \( \cup \), \( \cap \), \( \setminus \) and \( A \), \( B \), \( C \), \( D \) write down the sets of all \( x \) which satisfy:
   i) \( \{ x \in A \text{ or } x \in B \} \cap x \in C \setminus x \notin D \); ii) \( x \in A \text{ or } \{ x \in B \land x \in C \} \land x \notin D \).

b) Write as a union of disjoint intervals the sets of the real numbers \( x \in \mathbb{R} \) satisfying:
   i) \( \{ x < 20 \land x^2 < 100 \} \lor x \notin (-\infty,-1] \); ii) \( x < 20 \land \{ x^2 < 100 \text{ or } x \notin (-\infty,-1] \} \).

(* Does the place of the parentheses matter?

4) Let \( A \), \( B \) be sets. Answer the following:

a) \( \exists f : A \rightarrow B \) injective iff \( \exists g : B \rightarrow A \) surjective.

b) \( f : A \rightarrow B \) is injective iff \( \exists g : B \rightarrow A \) surjective satisfying \( g(f(x)) = x \) for all \( x \in A \).

c) \( f : A \rightarrow B \) is surjective iff \( \exists g : B \rightarrow A \) injective satisfying \( f(g(y)) = y \) for all \( y \in B \).

Cardinality of sets. Recall that the cardinality of a set \( A \), denoted by \( |A| \), is, intuitively, a kind of size of \( A \). Recall that \( |A| \leq |B| \) \( \text{Def} \Rightarrow \exists f : A \rightarrow B \) injective, and that one has:

**Theorem** (Cantor, Bernstein, Schroeder). \( |A| \leq |B| \text{ and } |B| \leq |A| \text{ if and only if there exists } f : A \rightarrow B \) bijective.

**Definition.** Let \( X \) be a set. Recall the following:

- \( X \) is called finite of cardinality \( |X| = n \geq 0 \), if either \( X = \emptyset \), and then \( |X| := 0 \), or \( \exists f : \{1, \ldots, n\} \rightarrow X \) bijective, thus \( X = \{x_1, \ldots, x_n\} \), where \( x_i := f(i) \), \( 1 \leq i \leq n \).
- \( X \) is called countable, if \( |X| = |\mathbb{N}| \), i.e., there exists a bijection \( f : X \rightarrow \mathbb{N} \).
- \( X \) is called at most countable, if \( |X| \leq |\mathbb{N}| \).
5) Let \( X \) be a non-empty set. Prove/disprove the following:
   a) If \( X \) is finite, then every injective (resp. surjective) map \( f : X \to X \) is bijective.
   b) If \( X \) is infinite, there exists injective (surjective) \( f : X \to X \) which are not bijective.

6) Let \( X \) be an arbitrary set, and \( \mathcal{P}(X) := \{ A \mid A \subseteq X \} \) be the power set of \( X \). Prove:
   a) If \( |X| = n \) is finite, then \( |\mathcal{P}(X)| = 2^n \).
   b) One has always: \( |X| < |\mathcal{P}(X)| \). Deduce from this that \( |\mathbb{N}| < |\mathbb{R}| \).

   [Hint to the second part of b): Define \( f : \mathcal{P}(\mathbb{N}) \to \mathbb{R} \) by \( f(A) := a_0a_1a_2\ldots a_n\ldots \) for \( A \subseteq \mathbb{N} \), where \( a_n = 1 \) if \( n \in A \), and \( a_n = 0 \) if \( n \notin A \). Then \( A \neq A' \) implies \( x_A \neq x_{A'} \) (WHY), hence \( f \) is injective, etc...]

7) Let \( X, Y \) be finite sets, say \( |X| = m \) and \( |Y| = n \). Prove/disprove the following assertions:
   a) \( |X \cup Y| + |X \cap Y| = |X| + |Y| \). What is the corresponding assertion for \( |X \cup Y \cup Z| \)?
   b) \( |X \times Y| = |X| \cdot |Y| \). What is the corresponding assertion for \( |X \times Y \times Z| \)?

8) Let \( A, B, A_n, n \in \mathbb{N} \) be at most countable sets. Prove the following assertions:
   a) \( A \times B \) is at most countable. Is the same true for \( A_0 \times \cdots \times A_n \) for all \( n \in \mathbb{N} \).
   b) \( A \cup B \) is at most countable. Is the same true for \( \bigcup_{n \in \mathbb{N}} A_n \)?
   c)* Is the same true for the (infinite cartesian) product \( A_0 \times \cdots \times A_n \times \ldots \)?

9) Complete the proof of the assertions from the class:
   a) + and \( \cdot \) in \( \mathbb{N} \) are associative, commutative, and 0, respectively 1 are neutral elements. Further, \( \cdot \) is distributive w.r.t to +
   b) + and \( \cdot \) have cancelation property in \( \mathbb{N} \), respectively \( \mathbb{N}_{>0} \), i.e., for \( n, m, k \in \mathbb{N} \) one has:
      - \( m + k = n + k \Rightarrow m = n \).
      - \( m \cdot k = n \cdot k \Rightarrow m = n \), provided \( k > 0 \).

10) Recall that for \( m, n \in \mathbb{N} \), one says that \( m \leq n \) if there exists \( k \in \mathbb{N} \) such that \( n = m + k \). Complete the proof of the assertions from the class:
    \( \leq \) is compatible with + and \( \cdot \).

That is, for all \( n, m, k \in \mathbb{N} \) one has:
   a) \( m \leq n \) iff \( m + k \leq n + k \).
   b) \( m \leq n \) iff \( m \cdot k \leq n \cdot k \), provided \( k > 0 \).