Math 202 / Problem Set 4 (two pages)

1) A relation $R$ on a set $A$ is called a quasi-ordering, if $R$ is reflexive and transitive (but not necessarily anti-symmetric). For a quasi-ordering $R$ on $A$, define a relation $\sim_R$ on $A$ by $x \sim_R y \overset{\text{def}}{=} xRy \& yRx$. Prove the following:
   a) $\sim_R$ is an equivalence relation on $A$, and let $\hat{A} := A/\sim_R$ be the set of equivalence classes.
   b) Define a relation $\leq$ on $\hat{A}$ by $\hat{x} \leq \hat{y} \overset{\text{def}}{=} xRy$. Then $\leq$ is a (well defined) ordering on $\hat{A}$.
   (* What is the converse to a), b) above?

2) Let $X$ be a non-empty set, and recall the symmetric difference $A \Delta B := (A \setminus B) \cup (B \setminus A)$ on $\mathcal{P}(X)$. Prove/disprove the following:
   a) The difference $A \setminus B$ on $\mathcal{P}(X)$ is not associative/commutative/has no neutral element.
   b) $\mathcal{P}(X), \Delta$ is an abelian group.

3) In the notation from Problem 2) above, prove/answer the following:
   a) $\mathcal{P}(X), \Delta, \cap$ is a commutative ring.
   b) Which elements in the ring $\mathcal{P}(X), \Delta, \cap$ are invertible?

4) Prove/answer the following:
   a) There exist $\sigma, \tau \in S_3$ such that $(\sigma \cdot \tau)^2 \neq \sigma^2 \cdot \tau^2$.
   b) Solve the equations $x \circ \begin{pmatrix} 1 2 3 \\ 2 3 1 \end{pmatrix} = \begin{pmatrix} 1 2 3 \\ 2 3 1 \end{pmatrix}$ and $\begin{pmatrix} 1 2 3 \\ 2 3 1 \end{pmatrix} \circ x = \begin{pmatrix} 1 2 3 \\ 2 1 3 \end{pmatrix}$ in $S_3$.
   c) Find the smallest $n_G$ such that $g^{n_G} = e_G$ for all $g \in G$, in the following cases:
      i) $G = S_3$; ii) $G = S_5$.

5) Denote i) $A_1A_2A_3$ triangles; ii) $B_1B_2B_3B_4$ quadrangles; iii) $C_1C_2C_3C_4C_5$ pentagons. Depending of further properties of these shapes, write in each case the group of transformations as permutation groups of the vertices. [Hence the results will be subgroups of $S_3$, $S_4$, $S_5$, respectively [WHY]. Recall what we did/said in class, e.g., in the case of triangles $A_1A_2A_3$, the group is $S_1, S_2$, or $S_2$ [WHY], etc.]

The monoid/group/ring of functions

Let $X, T$ be non-empty sets, and $\text{Maps}(X, T) := \{f \mid f : X \to T \text{ abstract map}\}$. Given a composition law $\cdot$ on $T$, define the *operation* on $\text{Maps}(X, T)$ by $(f \cdot g)(x) := f(x) \cdot g(x)$.

6) Prove/disprove the following assertions about the composition law $\cdot$ defined above:
   a) $\cdot$ is associative, reps. commutative iff $\cdot$ is so.
   b) $\cdot$ has a neutral element $e$ iff $\cdot$ has a neutral element $e_\cdot$. What is $e_\cdot$ as a function?
   c) $f \in \text{Maps}(X, T)$ is invertible w.r.t. $\cdot$ iff $f(x) \in T$ is invertible w.r.t. $\cdot$ for all $x \in X$.

7) In the context of Problem 4) above, prove or disprove the following:
   a) $G, \cdot$ is an (abelian) monoid, reps. group iff $\text{Maps}(X, G), \cdot$ is so.
   b) $R, +, \cdot$ is a (commutative) ring with $1_R$ iff the corresponding $\text{Maps}(X, R), \uplus, \cdot$ is so.
   (●) Question: Is the same true for (skew) fields $R, +, \cdot$ ?

Language: $\text{Maps}(X, T)$ is called the monoid/group/ring of $T$-valued maps on $X$. 

Due: Th, Oct 5, 2017 (in class)
(Cartesian) products of algebraic structures

[The product of monoids/groups/rings/(skew) fields]

Let $*$' and $*$'' be composition laws on $X'$, respectively $X''$. Define the coordinate wise composition law $*: = *' \times *''$ on $X := X' \times X''$ by $(x', x'') \ast (y', y'') := (x' \ast' y', x'' \ast'' y'')$.

8) Prove/disprove:
   a) $*$ is associative, reps. commutative if and only if $*$' and $*$'' are so.
   b) $*$ has a neutral element $e$ iff $*$' and $*$'' have neutral elements $e'$, $e''$.
   c) $x := (x', x'')$ is invertible iff $x'$ and $x''$ are invertible.

9) Let $G := G' \times G''$, $R := R' \times R''$ and $\ast = \ast' \times \ast''$, $\circ = \circ' \times \circ''$. Prove the following:
   1) $G'$, $\ast'$ and $G''$, $\ast''$ are (abelian) monoids, resp. groups, iff $G$, $\ast$ is so.
   2) $R'$, $\ast'$, $\circ'$ and $R''$, $\ast''$, $\circ''$ are (commutative) rings iff $R$, $\ast$, $\circ$ is so.

   (●) Question: Is the same true for fields $R'$, $R''$?

Miscellaenia:

10) For a commutative ring $R$, $+$, $\cdot$ and $a, b, c, d \in R$, using $+$ and $\cdot$ define on $R$ a new “addition” by $x \oplus y = x + y + a$ and a new “multiplication” by $x \odot y = xy + bx + cy + d$.
   a) Let $R = \mathbb{Z}$ or $R = \mathbb{R}$. Find all $a, b, c, d$ such that $\mathbb{Z}$, $\oplus$, $\cdot$ and/or $\mathbb{R}$, $\oplus$, $\cdot$ are rings.
   b) Solve in the ring $R$, $\oplus$, $\cdot$ the equations $x^2 \oplus 3 \cdot x = 1_R$ and $x^2 \oplus 3 \cdot x = 0_R$.

11) Let $R$ be a commutative ring with $1_R \neq 0_R$, and define the quaternions $\mathbb{H}_R$ over $R$ by: $\mathbb{H}_R := R^4 := \{a + b \imath + c \jmath + d \kappa \mid a, b, c, d \in R\}$ endowed with the coordinate wise addition $+$ and the multiplication $\cdot$ defined by: $\imath^2 = \jmath^2 = \kappa^2 = -1_R$, $\imath \cdot \jmath = \kappa$, $\jmath \cdot \kappa = \imath$, $\kappa \cdot \imath = \jmath$.
   a) Show that $\mathbb{H} := \mathbb{H}_R$ is a skew field. And solve the equation $(1 + \imath + \jmath + \kappa) \cdot x = x \cdot \imath$.
   b) Show that $\mathbb{H}_R, +, \cdot$ is a ring with $1_{\mathbb{H}_R}$, and $\mathbb{H}_R, +, \cdot$ is commutative iff $1_R = -1_R$. 

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