Recall that $\mathbb{Z} =: \mathbb{Z}/\sim := \mathbb{N} \times \mathbb{N}_{>0}/\sim$ defined as in the class, and recall the definitions of the addition $+$ and the multiplication $\cdot$ on $\mathbb{Z}$. Further, recall the definition of $\leq$ on $\mathbb{Z}$, and that for $a \in \mathbb{Z}$ we define the absolute value $|a|$ of $a$ to be $|a| = a$ if $a \geq 0$, $|a| = -a$ if $a \leq 0$.

1) Complete the proof of the assertions made in the class:
   a) $\cdot$ is associative, and distributive w.r.t. $+$
   b) The multiplication by non-zero $a \in \mathbb{Z}$ has cancellation.

2) $\mathbb{Z}, +, \cdot$ has no proper subrings, i.e., if $X \subset \mathbb{Z}$ is closed with respect to $+$ and $\cdot$ and $X, +, \cdot$ is a ring with neutral element $1_X$ for $\cdot$, then $X = \mathbb{Z}$.

3) Complete the proof of the assertions made in the class:
   a) The ordering $\leq$ is compatible with multiplication and has cancellation by positive numbers $a \in \mathbb{Z}$, $a > 0$.
   b) For every $a \in \mathbb{Z}$ one has $a^2 \geq 0$, and there exist a unique $n \in \mathbb{N}$ such that $a^2 = n^2$. If so, then $n = |a|$.

4) Show that one has division with remainder in $\mathbb{Z}$, i.e., for $a, b \in \mathbb{Z}$, $b \neq 0$ there exist unique $q \in \mathbb{Z}$, $r \in \mathbb{N}$ such that
   $$a = b \cdot q + r, \quad 0 \leq r < |b|.$$
[Hint. Intuitively, \( \sup(X) = \sqrt{2} \), but \( \sqrt{2} \notin \mathbb{Q} \) (WHY). Now make this argument mathematically correct (HOW), etc.]

**Generalities**

9) Let \( S_5 \) be the permutations group of \( \{1, 2, 3, 4, 5\} \), and consider \( \sigma = (1 2 3 4 5) \) and \( \tau = (1 2 3 4 5) \) as elements of \( S_5 \). Solve the following equations in \( S_5 \) for the unknown \( x \):

- a) \( x \circ \sigma = \tau \).
- b) \( x \circ x \circ \sigma = \tau \), respectively \( x \circ \sigma \circ x = \tau \).

10) Show that the following equations in the unknown \( x \) have no solutions in \( \mathbb{Q} \).

a) \( x^7 = 15 \).

b) \( x^n = a \), where \( a \in \mathbb{Q}, a \neq 0, 1 \), and \( n \in \mathbb{N}_{>0} \) sufficiently large.