Due: Th, Oct 19, 2017 (in class)

Math 202 / Problem Set 5 (two pages)

Recall that $\mathbb{Z} =: \mathcal{Z}/\!\sim := \mathbb{N} \times \mathbb{N}_{>0}/\!\sim$ defined as in the class, and recall the definitions of the addition $+$ and the multiplication $\cdot$ on $\mathbb{Z}$. Further, recall the definition of $\leqslant$ on $\mathbb{Z}$, and that for $a \in \mathbb{Z}$ we define the absolute value $|a|$ of a to be $|a| = a$ if $a \geq 0$, $|a| = -a$ if $a \leq 0$.

1) Complete the proof of the assertions made in the class:

a) $\cdot$ is associative, and distributive w.r.t. $+$

b) The multiplication by non-zero $a \in \mathbb{Z}$ has cancellation.

2) $\mathbb{Z}, +, \cdot$ ***has no proper subrings***, i.e., if $X \subset \mathbb{Z}$ is closed with respect to $+$ and $\cdot$ and $X, +, \cdot$ is a ring with neutral element 1_X for $\cdot$, then $X = \mathbb{Z}$.

3) Complete the proof of the assertions made in the class:

a) The ordering $\leqslant$ is compatible with multiplication and has cancellation by positive numbers $a \in \mathbb{Z}$, $a > 0$.

b) For every $a \in \mathbb{Z}$ one has $a^2 \geqslant 0$, and there exist a unique $n \in \mathbb{N}$ such that $a^2 = n^2$. If so, then $n = |a|$.

4) Show that one has division with remainder in $\mathbb{Z}$, i.e., for $a, b \in \mathbb{Z}$, $b \neq 0_{\mathbb{Z}}$ there exist unique $q \in \mathbb{Z}$, $r \in \mathbb{N}$ such that

$$a = b \cdot q + r, \quad 0 \leq r < |b|.$$

Recall that $\mathbb{Q} =: \mathcal{Q}/\!\sim := \mathbb{Z} \times \mathbb{Z}^{\bullet}/\!\sim$ defined as in the class, and recall the definitions of the addition $+$ and the multiplication $\cdot$ on $\mathbb{Q}$. Further, recall that every rational number a/s has a unique representative of the form a_0/r_0, where $r_0 \in \mathbb{N}_{>0}$, $a = n$ or $a = -n$, with $n \in \mathbb{N}$ and a, r_0 relatively prime. Finally, recall that $a/r \geqslant 0$ if, by definition, a_0/r_0 satisfies $a_0 \geqslant 0$ in $\mathbb{Z}$. Complete the proof of the assertions made in the class:

5) Addition $+$ and multiplication $\cdot$ in $\mathbb{Q}$ are well defined and $\mathbb{Q}, +, \cdot$ is a field (what does that mean?).

6) $\mathbb{Q}, +, \cdot$ ***has no proper subfields***, i.e., if a subset $X \subset \mathbb{Q}$ is closed with respect to addition and multiplication, and $X, +, \cdot$ is a field, then $X = \mathbb{Q}$.

7) Complete the proof of the assertions made in the class:

a) For $a/r \in \mathbb{Q}$ one has: $a/r \geq 0$ iff either $a = 0$ or $a, r \neq 0$ have the same sign in $\mathbb{Z}$.

b) The ordering $\leqslant$ on $\mathbb{Q}$ has the properties:

- $\leqslant$ is a total ordering.
- $\leqslant$ is compatible with $+$
- $\leqslant$ is compatible with $\cdot$ and has cancelation w.r.t. positive numbers $x > 0$.

8) Prove/answer the following:

a) $\mathbb{Q}$ has no isolated elements, i.e., $\forall\, a, b \in \mathbb{Q}$, $a \neq b$, $\exists\ c \in \mathbb{Q}$ s.t. $a < c < b$ or $b < c < a$.

b) In particular, "most of the subsets" $X \subset \mathbb{Q}$ are not well ordered (WHY).

c) Let $X := \{a \in \mathbb{Z} \mid a^2 < 2\} \subset \mathbb{Q}$. Show that $\sup(X)$, $\inf(X)$ do not exist in $\mathbb{Q}$.

[**Hint.** Intuitively, $\sup(X) = \sqrt{2}$, but $\sqrt{2} \notin \mathbb{Q}$ (WHY). Now make this argument mathematically correct (HOW), etc.]

Generalities

9) Let S_5 be the permutations group of $\{1, 2, 3, 4, 5\}$, and consider $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$ as elements of S_5. Solve the following equations in S_5 for the unknown x:

- a) $x \circ \sigma = \tau$.
- b) $x \circ x \circ \sigma = \tau$, respectively $x \circ \sigma \circ x = \tau$.

10) Show that the following equations in the unknown x have no solutions in $\mathbb{Q}$.

a) $x^7 = 15$.

b) $x^n = a$, where $a \in \mathbb{Q}$, $a \neq 0, 1$, and $n \in \mathbb{N}_{>0}$ sufficiently large.