Due: Th, Nov 2, 2017 (in class)

Math 202 / Problem Set 7 (two pages)

Recall that for any set X, the set of (infinite) sequences with values in X is, by definition, the set $\mathrm{Maps}(\mathbb{N}, X)$, and for $\underline{\boldsymbol{x}} : \mathbb{N} \to X$ we denote $\underline{\boldsymbol{x}} = (x_n)_n$, where $x_n : -\underline{\boldsymbol{x}}(n)$ for all $n \in \mathbb{N}$.

Further we set $\mathcal{S} := \mathcal{S}(\mathbb{Q})$, and denote by $\mathcal{C} := \mathcal{C}(\mathbb{Q})$, $\mathcal{R} := \mathcal{S}(\mathbb{Q})$ the sets of convergent sequences, respectively Cauchy sequences with values in $\mathbb{Q}$. In particular, one has $\mathcal{C}, \mathcal{R} \subset \mathcal{S}$ (WHY).

1) Let $(x_n)_n, (y_n)_n \in \mathcal{S}$ be given. Prove/disprove:

a) If $(x_n)_n + (y_n)_n$ is convergent, so are $(x_n)_n$ and $(y_n)_n$. The same question, if $(x_n)_n$ or $(y_n)_n$ is convergent.

b) The same questions concerning Cauchy sequences.

c) The same questions about the product $(x_n)_n \cdot (y_n)_n$ of the sequences $(x_n)_n$ and $(y_n)_n$.

2) True or false (**justify** your answer, that is, prove/disprove the assertion):

a) Suppose that $x_n \neq 0$ for all n, and $(1/x_n)_n$ is convergent. Is $(x_n)_n$ convergent?

b) If $(x_n)_n$ is a sequence such that $(x_n^3 + x_n)_n$ is convergent, then $(x_n)_n$ is convergent. Same questions for Cauchy sequences.

c) Let $f(t)$ be a polynomial in the variable t with rational coefficients. If $(x_n)_n$ is convergent, so is the sequence $(y_n)_n$, where $y_n = f(x_n)$ for all n.
Same question for Cauchy sequences.

An abstract Lemma

Let $X, \leqslant$ be a totally ordered set, and $(x_n)_n$ be a sequence with values in X. Then one of the following two cases occur (WHY):

Case 1) For all $n \in \mathbb{N}$, one has: $\max\{x_{n'} \mid n' \geqslant n\}$ exists.

Case 2) There exists $n_0 \in \mathbb{N}$ such that $\max\{x_{n'} \mid n' \geqslant n_0\}$ does not exist.

3) Complete the proof of the following assertion from the class: Suppose that $(x_n)_n$ is as in Case 1) above. Setting $y_n := \max\{x_{n'} \mid n' \geqslant n\}$ for all $n \in \mathbb{N}$, prove the following:

a) $(y_n)_n$ is decreasing, and $x_n \leq y_n$ for all n

b) $(x_n)_n$ and (y_n) have a common subsequence. What is that subsequence?

Next suppose that $(x_n)_n$ are as in Case 2) above. Then there exists a minimal $n_1 > n_0$ such that $x_{n_1} > x_{n_0}$, and by induction: For every $k \geqslant 0$, there exists $n_{k+1} > n_k$ minimal with the property $x_{n_{k+1}} > x_{n_k}$ (WHY).

4) Complete the proof of the following assertion from the class: Suppose that $(x_n)_n$ is as in Case 2) above, and in the previous notation, set $z_m = x_{n_0}$ for $m < n_0$, and further: $z_n := x_{n_k}$ for $n_k \leq n < n_{k+1}$. [Hence $z_n = x_{n_0}$ for $n_0 \leqslant n < n_1$, $z_n = x_{n_1}$ for $n_1 \leqslant n < n_2$, etc.] Prove the following:

a) $(z_n)_n$ is increasing, and $x_n \leq z_n$ for all $n \geqslant n_0$.

b) $(x_n)_n$ and (z_n) have a common subsequence. What is that subsequence?

5) Prove the assertions from the class:

a) For every Cauchy sequence $(x_n)_n$ there exist increasing, respectively decreasing sequences $(x'_n)_n$, respectively $(x''_n)_n$ such that $(x'_n)_n \sim (x_n)_n \sim (x''_n)_n$.

b) Are there such sequences which are strictly increasing, respectively decreasing?

[**Hint.** To a): Use problems 3), 4) above. To b): Prove and use the following fact: If $(z_n)_n$ is decreasing, then $(z_n)_n + (1/n)_n$ is strictly decreasing (WHY), and $(z_n)_n \sim \big((z_n)_n + (1/n)_n\big)_n$ (WHY). Similarly for increasing sequences...]

6) Complete the proof of the assertions from the class:

a) Multiplication in $\mathbb{R}$ is compatible with the ordering $\leqslant$.

b) For $x \in \mathbb{R}$ one has: $x \geqslant 0_{\mathbb{R}}$ iff $\exists\, x_0 \in \mathbb{R}$ such that $x = x_0^2$.

7 Prove in all detail the following assertions from the class:

$\mathbb{Q}$ *is dense in* $\mathbb{R}$, *i.e., if* $x, y \in \mathbb{R}$ *and* $x < y$, *then there exist* $a \in \mathbb{Q}$ *s.t.* $x < a < y$.

[**Hint.** Let $(x'_n)_n$ be decreasing and $(x_n)_n \sim (x'_n)_n$ and $(y'_n)_n$ be increasing and $(y_n)_n \sim (y'_n)_n$; such Cauchy sequences exist (WHY). Then $\exists\, N$ such that $y'_n - x'_n > 0$ for $n \geqslant N$ (WHY), and setting $a := (y'_N + x'_N)/2$ one has: $x'_n < a < y'_n$ for $n > N$ (WHY), hence $x < a < y$ (WHY).]

Have fun!

8) Prove/disprove/answer the following:

a) Every real number $x \in \mathbb{R}$ has a *decimal representation*, i.e., there exist $(a_n)_n$ with $a_0 \in \mathbb{N}$, $0 \leqslant a_n < 10$ for $n > 0$ such that the sequence $(x_n)_n$, $x_n = \sum_{i=0}^{n} \frac{a_i}{10^i}$ is a Cauchy sequence representing x, provided $x \geqslant 0$. What is to be done if $x < 0$?

(∗) Is the sequence $(a_n)_n$ unique for all / some real numbers $x \in \mathbb{R}$?

b) Show that the same holds with 10 replaced by any other basis $k > 1$, e.g. by $k = 2$ (leading to the so called *binary representation* of numbers — used in computers).

c) Let $(x_n)_n$ be a Cauchy sequence of positive numbers such that the sequence $(x_n^n)_n$ is bounded away from zero. Show that $x_n \to 1$. Is the converse true?

Continuous fraction expansion (optional) [Google it!]

9)[**] Let $x \in \mathbb{R}$ be a fixed real number. Prove the following:

a) There exists a unique integer $n_0 \in \mathbb{Z}$ such that $n_0 \leqslant x < n_0 + 1$.

Terminology. n_0 is called the integer part of x, denoted $\lfloor x \rfloor$.

b) Set $y_0 := x$, and $z_1 := y_0 - \lfloor y_0 \rfloor$. If $z_1 = 0$, **STOP**. If not, show that $0 < z_1 < 1$, **DO**:

c) Set $y_1 := 1/z_1$, $n_1 := \lfloor y_1 \rfloor$, $z_2 := y_1 - \lfloor y_1 \rfloor$. If $z_2 = 0$, **STOP**. If not, $0 < z_2 < 1$, **DO**:

d) Set $y_2 := 1/z_2$, $n_2 := \lfloor y_2 \rfloor$, $z_3 := y_2 - \lfloor y_2 \rfloor$. If $z_3 = 0$, **STOP** ...

Proceed by induction, and get a possibly finite sequence of natural numbers $(n_k)_k$.

For every k such that $n_0, n_1, n_2, \dots, n_k$ are defined, set

$$x_0 := n_0, \quad x_1 = n_0 + \frac{1}{n_1}, \quad x_2 = n_0 + \frac{1}{n_1 + \frac{1}{n_2}}, \quad \dots, \quad x_k = n_0 + \cfrac{1}{n_1 + \cfrac{1}{\ddots \, \cfrac{}{n_{k-1} + \frac{1}{n_k}}}}$$

e) $(x_k)_k$ is a finite sequence iff $x \in \mathbb{Q}$. If $x \notin \mathbb{Q}$, then $(x_k)_k$ is convergent, and $x_k \to x$.