Math 314 / Midterm 1 (two pages)

Academic Integrity Statement:
My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this Math 314 exam. [That means, among other things, that you are allowed to: (a) discuss the problems of the midterm with your colleagues, but not work out solutions together; (b) ask any member the Math Dept about hints to the exam problems, but you must first mention to her/him that it goes about problems on a take home exam; (c) consult resources (books, Internet, etc.), but you must design/write down your own proofs.]

Name (printed): __________________________ Signature: __________________________

Note: There are 8 (eight) problems on this exam.
Points: 12.5 points for each problem (extra and/or partial credit possible).
Grading: 50 < C−, C, C+ < 70 < B−, B, B+ < 85 < A−, A, A+

Procedures: Print out the two pages of the exam and staple them to your work. Write your name (printed) and sign the above Academic Integrity Statement.

• Recall: A complete proof must contain all the necessary explanations/steps, and in order to disprove an assertion you must give a counterexample showing that the assertion is not true.

1) Let $f : G \to H$ be a group homomorphism, and $G' \subset G$, $H' \subset H$ be subsets —note: not necessarily subgroups! Prove/disprove the following:
   a) $f(G') \subset H$ is a subgroup iff $G' \subset G$ is a subgroup. Same question if $f$ is injective.
   b) $f^{-1}(H') \subset G$ is a subgroup iff $H' \subset H$ is a subgroup. Same question if $f$ is surjective

2) Prove/disprove/answer the following:
   a) Let $M := \{ f : \mathbb{R} \to \mathbb{R} \mid \exists a, b \in \mathbb{R} \text{ s.t. } f(x) = ax + b \forall x \in \mathbb{R} \}$. Then $M$ endowed with the usual composition of maps $\circ$ is a monoid. Is $M$ commutative? For which values of $a, b \in \mathbb{R}$ is $f(x) = ax + b$ and invertible element of $M$?
   b) For every $\sigma, \tau \in S_5$, the equation $x \circ \sigma = x^{-1} \circ \tau$ has solutions in $S_5$.

3) Let $R_1, R_2$ be rings with $1_{R_1}, 1_{R_2}$, and let $R = R_1 \times R_2$ be the product ring with the component wise addition and multiplication. Prove/disprove the following:
   a) For $x = (x_1, x_2) \in R$ with $x_1 \in R_1$, $x_2 \in R_2$, one has: $x$ is a zero divisor iff both $x_1$ and $x_2$ are zero divisors. And $x = (x_1, x_2)$ is invertible iff both $x_1$ and $x_2$ are invertible.
   b) Suppose that $R$ is finite. Then for every $x_1 \in R_1$ one has: If $x_1$ is not a zero divisor, then $x_1$ is invertible.
4) Let \( R = \mathbb{Z}/120\mathbb{Z} \), and \( M \) be an \( R \)-module. Prove/disprove/answer the following:
   a) Let \( x \in M \) be given. Then \( x = 0_M \iff 35 \cdot x = 0_M \) and \( 56 \cdot x = 0_M \).
   b) Let \( u, v, w \in M \) be linearly independent. For which values of \( a, b \in R \), the elements \( au + bv + (a + b)w \) and \( a^2u + b^2v + (a^2 + b^2)w \) are linearly dependent?

5) Recall that \( A \in R^{m \times m} \) is called nilpotent, if \( A^k = 0_{m \times m} \) for some \( k > 0 \). Prove/disprove:
   a) If \( A, B \in R^{2 \times 2} \) are nilpotent, so is \( A + B \) and \( A^2 + B^2 \). Same question, provided \( AB = BA \).
   b) There exist nilpotent matrices \( A \in R^{2 \times 2} \) such that \( A^3 \neq 0_{2 \times 2} \).

6) Let \( V \subset \text{Maps}(\mathbb{R}, \mathbb{R}) \) be the subspace spanned by \( f_1 = x \sin 2x \), \( f_2 = x \cos 2x \), \( f_3 = \sin 2x \), \( f_4 = \cos 2x \), \( f_5 = 1 \). Prove that \( T : V \rightarrow V \) by \( T(f) = f''' - f \) is a linear transformation.
   a) Find a matrix \( A \in \mathbb{R}^{5 \times 5} \) such that \( (Tf_1, \ldots, Tf_5) = (f_1, \ldots, f_5)A \).
   b) Prove that \( f_1, f_2, f_3, f_4, f_5 \) is a basis of \( V \), and describe \( \ker(T) \) and \( \text{im}(T) \).

7) Using elementary matrices, find/define the relations between the following matrices:
   \[
   A := \begin{pmatrix}
   a_1 & a_2 & a_3 \\
   b_1 & b_2 & b_3 \\
   c_1 & c_2 & c_3
   \end{pmatrix} \quad B := \begin{pmatrix}
   a_1 & 2a_2 & 5a_3 \\
   b_1 & 2b_2 & 5b_3 \\
   3c_1 & 6c_2 & 15c_3
   \end{pmatrix} \quad C := \begin{pmatrix}
   a_1 & 4a_2 + 6a_1 & -2a_3 \\
   \frac{1}{2}b_1 & 2b_2 + 3b_1 & -b_3 \\
   \frac{1}{2}c_1 & 8c_2 + 12c_1 & -4c_3
   \end{pmatrix}
   \]

8) Discuss the existence and uniqueness of the solution \( (t, x, y, z) \) of the system of equations below, where \( a, b, c, d, e \) are arbitrary elements of a field \( F \) (the answer might depend on \( F \)):
   \[
   \begin{cases}
   at - x + 2y + z = -c \\
   -y + bz = c \\
   x - y = c \\
   t + x + y + z = abd
   \end{cases}
   \]