Math 314 / Problem Set 2 (two pages)

Basics

- Let $A, B, C, D$ be given sets, and $x$ be elements, e.g., real numbers. Answer the following:
  a) Using $\cup, \cap, \setminus$ and $A, B, C, D$ write down the sets of all $x$ which satisfy:
    i) $(x \in A \text{ or } x \in B) \& x \in C \& x \notin D$; ii) $x \in A$ or $(x \in B \& x \in C) \& x \notin D$.
  b) Write as a union of disjoint intervals the sets of the real numbers $x \in \mathbb{R}$ satisfying:
    i) $(x < 20 \& x^2 < 100)$ or $x \notin (-\infty, -1]$; ii) $x < 20 \& (x^2 < 100 \text{ or } x \notin (-\infty, -1])$.
  
  (*) Does the place of the parentheses matter?

- Answer the following:
  a) $\exists f : A \to B$ injective iff $\exists g : B \to A$ surjective.
  b) $f : A \to B$ is injective iff $\exists g : B \to A$ surjective satisfying $g(f(x)) = x \forall x \in A$.

Cardinality of sets. The cardinality of a set $A$, denoted by $|A|$, is intuitively a kind of size of $A$. Define: $|A| \leq |B| \iff \exists f : A \to B$ injective. Recall the following famous difficult:

**Theorem.** $|A| \leq |B|$ and $|B| \leq |A|$ if and only if there exists $f : A \to B$ bijective.

Further, $X$ is called finite of cardinality $|X| = n \geq 0$, if either $X = \emptyset$, and then $|X| := 0$, or $\exists f : \{1, \ldots, n\} \to X$ bijective, thus $X = \{x_i := f(i) \mid i = 1, \ldots, n\} = \{x_1, \ldots, x_n\}$.  

1) Let $X$ be a non-empty set. Prove/disprove the following:
   a) If $X$ is finite, then every injective (resp. surjective) map $f : X \to X$ is bijective.
   b) If $X$ is infinite, there exists injective (surjective) $f : X \to X$ which are not bijective.

2) Let $X$ be an arbitrary set, and $\mathcal{P}(X) := \{A \mid A \subseteq X\}$ be the power set of $X$. Prove:
   a) If $|X| = n$ is finite, then $|\mathcal{P}(X)| = 2^n$.
   b) One has always: $|X| < |\mathcal{P}(X)|$. Deduce from this that $|\mathbb{N}| < |\mathbb{R}|$.

[HINT to the second part of b): Define $f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ by $f(A) := a_0a_1a_2\ldots$ for $A \subseteq \mathbb{N}$, where $a_n = 1$ if $n \in \mathbb{N}$, and $a_n = 0$ if $n \notin \mathbb{N}$. Then $N \neq N'$ implies $x_N \neq x_{N'}$ (WHY), hence $f$ is injective, etc...]

3) Let $X, Y$ be finite sets, say $|X| = m$ and $|Y| = n$. Prove/disprove the following assertions:
   a) $|X \cup Y| + |X \cap Y| = |X| + |Y|$. What is the corresponding assertion for $|X \cup Y \cup Z|$?
   b) $|X \times Y| = |X| \cdot |Y|$. What is the corresponding assertion for $|X \times Y \times Z|$?

Composition laws

4) Let $I = [0,1]$ or $I = (0,1)$, and $A \in \mathbb{R}^{2 \times 2}$ be $2 \times 2$ matrices. In each of the following cases determine the largest monoid, respectively group in the specified set:
   - The interval $I$ endowed with the usual addition, respectively multiplication.
   - $C^<$ := \{f : I \to I \mid f \text{ continuous increasing }\} w.r.t. $+$, resp. multiplication of functions.
   - The set of all the antisymmetric matrices $A \in \mathbb{R}^{2 \times 2}$ w.r.t. multiplication.
   - The set of all the matrices $A \in \mathbb{R}^{2 \times 2}$ having $\det(A) > 0$ w.r.t. multiplication.
- The set of matrices $A \in \mathbb{R}^{2 \times 2}$ having even integer coefficients w.r.t. addition.
- The set of matrices $A \in \mathbb{R}^{2 \times 2}$ with non-negative coefficients w.r.t. multiplication.
- The set of all the reflections about lines trough the origin w.r.t. composition of maps.
- $\mathcal{F}(X) := \{ f : X \to X \mid f \text{ arbitrary map } \}$ w.r.t. map composition.
- The unit circle $\mathbb{S} := \{ z \in \mathbb{C} \mid |z| = 1 \}$ w.r.t. multiplication.
- $S := \{ f : \mathbb{R} \to \mathbb{R} \mid f(x) = ax + b, a, b \in \mathbb{R}, a \neq 0 \}$ w.r.t. composition $\circ$ of maps.
- $\{2^n \mid m \in \mathbb{Z} \}$ w.r.t. multiplication.

5) Let $X$ be a non-empty set. The symmetric difference on $\mathcal{P}(X) := \{ A \mid A \subseteq X \}$ is defined by $A \Delta B := (A\setminus B) \cup (B\setminus A)$. Prove that $\mathcal{P}(X)$ endowed with $\Delta$ is an abelian group.

6) Let $S_5$ be the permutations group of $\{1, 2, 3, 4, 5\}$, and consider $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$ as elements of $S_5$. Solve the following equations in $S_5$ for the unknown $x$:
   a) $x \circ \sigma = \tau$.
   b) $x^2 \circ \sigma = \tau$, respectively $x \circ \sigma \circ x = \tau$.

7) Describe as permutation groups of the vertices of the following transformation groups:
   a) The transformation group $T_{ABCD}$ of the square $ABCD$.
   b) The transformation group of the regular pentagon $ABCDE$.
   c) The transformation group of the cube $ABCDAB'C'D'$ which fix the vertices $A, A'$.

8) Let $M, \ast$ be a monoid with neutral element $e \in M$. Prove/disprove the following:
   a) $G := \{ g \in M \mid x \text{ has an inverse in } M \}$ endowed with $\ast$ is a group.
   b) If for even $x \in M$ there exits $x' \in M$ such that $x \ast x = e$, then $M, \ast$ is a group.

9) Let $R, +, \cdot$ be a commutative ring with $0_R \neq 1_R$. Recall that the set of invertible elements of $R$ is $R^\times := \{ x \in R \mid x \text{ invertible w.r.t to multiplication} \}$. Prove/disprove:
   a) All $r \in R^\times$ are not zero divisors, i.e., for all $x \in R, x \neq 0_R$, one has that $rx \neq 0_R$.
   b) $R^\times$ is a group with respect to the multiplication.
   c) For every $r \in R$, the set $rR := \{ rx \mid x \in R \}$. Then one has:
      - $r_1R = r_2R$ if and only if there exists $x \in R^\times$ such that $r_2 = x r_1$.
      - $rR = R$ if and only if $r$ is invertible.

(Cartesian) products of algebraic structures

Let $\ast'$ and $\ast''$ be composition laws on $X'$, respectively $X''$. Define the coordinate wise composition law $\ast := \ast' \times \ast''$ on $X := X' \times X''$ by $(x', x'') \ast (y', y'') := (x' \ast' y', x'' \ast'' y'')$.

10) Prove/disprove:
   a) $\ast$ is associative, reps. commutative if and only if $\ast'$ and $\ast''$ are so.
   b) $\ast$ has a neutral element $e$ iff $\ast'$ and $\ast''$ have neutral elements $e', e''$.
   c) $x := (x', x'')$ is invertible iff $x'$ and $x''$ are invertible.

11) Let $G := G' \times G''$, $R := R' \times R''$ and $\ast = \ast' \times \ast''$, $\circ = \circ' \times \circ''$. Prove the following:
   1) $G', \ast'$ and $G'', \ast''$ are (abelian) monoids, resp. groups, iff $G, \ast$ is so.
   2) $R', \ast', \circ'$ and $R'', \ast'', \circ''$ are (commutative) rings iff $R, \ast, \circ$ is so.

(•) Question: Is the same true for fields $R', R''$?