Due: Mo, Sept 18, 2017

Math 314 / Problem Set 2 (two pages)

Basics

- Let A, B, C, D be given sets, and x be elements, e.g., real numbers. Answer the following:
  a) Using $\cup$, $\cap$, $\setminus$ and A, B, C, D write down the sets of all x which satisfy:
     i) $\Big(x \in A$ or $x \in B\Big)$ & $x \in C$ & $x \notin D$; ii) $x \in A$ or $\Big(x \in B$ & $x \in C\Big)$ & $x \notin D$.
  b) Write as a union of disjoint intervals the sets of the real numbers $x \in \mathbb{R}$ satisfying:
     i) $\Big(x < 20$ & $x^2 < 100\Big)$ or $x \notin (-\infty, -1]$; ii) $x < 20$ & $\Big(x^2 < 100$ or $x \notin (-\infty, -1]\Big)$.
  $(*)$ Does the place of the parentheses matter?

- Answer the following:
  a) $\exists\, f : A \to B$ injective iff $\exists\, g : B \to A$ surjective.
  b) $f : A \to B$ is injective iff $\exists\, g : B \to A$ surjective satisfying $g\Big(f(x)\Big) = x\ \forall\, x \in A$.

Cardinality of sets. The cardinality of a set A, denoted by $|A|$, is *intuitively* is a kind of size of A. Define: $|A| \leq |B| \overset{\text{Def}}{\Longleftrightarrow} \exists\, f : A \to B$ injective. Recall the following *famous difficult*:

Theorem. $|A| \leq |B|$ *and* $|B| \leq |A|$ *if and only if there exists* $f : A \to B$ *bijective.*

Further, X is called finite of cardinality $|X| = n \geqslant 0$, if either $X = \emptyset$, and then $|X| := 0$, or $\exists\, f : \{1, \ldots, n\} \to X$ bijective, thus $X = \{x_i := f(i) \,|\, i = 1, \ldots, n\} = \{x_1, \ldots, x_n\}$.

1) Let X be a non-empty set. Prove/disprove the following:

a) If X is finite, then every injective (resp. surjective) map $f : X \to X$ is bijective.
b) If X is infinite, there exists injective (surjective) $f : X \to X$ which are not bijective.

2) Let X be an arbitrary set, and $\mathcal{P}(X) := \{A \,|\, A \subseteq X\}$ be the power set of X. Prove:

a) If $|X| = n$ is finite, then $|\mathcal{P}(X)| = 2^n$.
b) One has always: $|X| < |\mathcal{P}(X)|$. Deduce from this that $|\mathbb{N}| < |\mathbb{R}|$.

[**Hint** to the second part of b): Define $f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ by $f(A) := \overline{a_0.a_1 \ldots a_n \ldots}$ for $A \subseteq N$, where $a_n = 1$ if $n \in N$, and $a_n = 0$ if $n \notin N$. Then $N \neq N'$ implies $x_N \neq x_{N'}$ (WHY), hence f is injective, etc...]

3) Let X, Y be finite sets, say $|X| = m$ and $|Y| = n$. Prove/disprove the following assertions:

a) $|X \cup Y| + |X \cap Y| = |X| + |Y|$. What is the corresponding assertion for $|X \cup Y \cup Z|$?
b) $|X \times Y| = |X| \cdot |Y|$. What is the corresponding assertion for $|X \times Y \times Z|$?

Composition laws

4) Let $I = [0,1]$ or $I = (0,1)$, and $A \in \mathbb{R}^{2\times 2}$ be 2×2 matrices. In each of the following cases determine the largest monoid, respectively group in the specified set:

- The interval I endowed with the usual addition, respectively multiplication.
- $\mathcal{C}^{\leqslant} := \{f : I \to I \,|\, f$ continuous increasing $\}$ w.r.t. $+$, resp. multiplication of functions.
- The set of all the antisymmetric matrices $A \in \mathbb{R}^{2\times 2}$ w.r.t. multiplication.
- The set of all the matrices $A \in \mathbb{R}^{2\times 2}$ having $\det(A) > 0$ w.r.t. multiplication.

- The set of matrices $A \in \mathbb{R}^{2\times 2}$ having even integer coefficients w.r.t. addition.
- The set of matrices $A \in \mathbb{R}^{2\times 2}$ with non-negative coefficients w.r.t. multiplication.
- The set of all the reflections about lines trough the origin w.r.t. composition of maps.
- $\mathcal{F}(X) := \{f : X \to X \mid f \text{ arbitrary map }\}$ w.r.t. map composition.
- The unit circle $\mathbb{S} := \{z \in \mathbb{C} \,\big|\, |z| = 1\}$ w.r.t. multiplication.
- $\mathcal{S} := \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = ax + b,\ a, b \in \mathbb{R},\ a \neq 0\}$ w.r.t. composition $\circ$ of maps.
- $\{2^m \mid m \in \mathbb{Z}\}$ w.r.t. multiplication.

5) Let X be a non-empty set. The symmetric difference on $\mathcal{P}(X) := \{A \mid A \subseteq X\}$ is defined by $A \vartriangle B := (A \backslash B) \cup (B \backslash A)$. Prove that $\mathcal{P}(X)$ endowed with $\vartriangle$ is an abelian group.

6) Let S_5 be the permutations group of $\{1, 2, 3, 4, 5\}$, and consider $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{smallmatrix}\right)$ and $\tau = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{smallmatrix}\right)$ as elements of S_5. Solve the following equations in S_5 for the unknown x:

a) $x \circ \sigma = \tau$.

b) $x^2 \circ \sigma = \tau$, respectively $x \circ \sigma \circ x = \tau$.

7) Describe as permutation groups of the vertices of the following transformation groups:

a) The transformation group $\mathcal{T}_{ABCD}$ of the square $ABCD$.

b) The transformation group of the regular pentagon $ABCDE$.

c) The transformation group of the cube $ABCDA'B'C'D'$ which fix the vertices A, A'.

8) Let $M, *$ be a monoid with neutral element $e \in M$. Prove/disprove the following:

a) $G := \{g \in M \mid x \text{ has an inverse in } M\}$ endowed with $*$ is a group.

b) If for even $x \in M$ there exits $x' \in M$ such that $x' * x = e$, then $M, *$ is a group.

9) Let $R, +, \cdot$ be a commutative ring with $0_R \neq 1_R$. Recall that the set of invertible elements of R is $R^\times := \{x \in R \mid x \text{ invertible w.r.t to multiplication}\}$. Prove/disprove:

a) All $r \in R^\times$ are not zero divisors, i.e., for all $x \in R$, $x \neq 0_R$, one has that $rx \neq 0_R$.

b) $R^\times$ is a group with respect to the multiplication.

c) For every $r \in R$, the set $rR := \{rx \mid x \in R\}$. Then one has:

- $r_1 R = r_2 R$ if and only if there exists $x \in R^\times$ such that $r_2 = x r_1$.
- $rR = R$ if and only if r is invertible.

(Cartesian) products of algebraic structures

Let $*'$ and $*''$ be composition laws on X', respectively X''. Define the coordinate wise composition law $* := *' \times *''$ on $X := X' \times X''$ by $(x', x'') * (y', y'') := (x' *' y',\ x'' *'' y'')$.

10) Prove/disprove:

a) $*$ is associative, reps. commutative if and only if $*'$ and $*''$ are so.

b) $*$ has a neutral element e iff $*'$ and $*''$ have neutral elements e', e''.

c) $x := (x', x'')$ is invertible iff x' and x'' are invertible.

11) Let $G := G' \times G''$, $R := R' \times R''$ and $* = *' \times *''$, $\circ = \circ' \times \circ''$. Prove the following:

1) *$G', *'$ and $G'', *''$ are (abelian) monoids, resp. groups, iff $G, *$ is so.*

2) *$R', *', \circ'$ and $R'', *'', \circ''$ are (commutative) rings iff $R, *, \circ$ is so.*

($\bullet$) **Question:** Is the same true for fields R', R'' ?