Math 114: Typical Problems Sections 12, 13, 14

1. Find the point in the plane tangent to the surface $-x + 2y^2 + z^2 = 3$ at $(0, 1, 1)$ which is closest to the point $(0, 2, 4)$.

2. Let $\mathcal{P}_1$ be the plane tangent to the surface $z = x^2 + y^2 - e^{x+y} - \sin(yx)$ at the point defined by $(x, y) = (0, 0)$, and $\mathcal{P}_2$ be the plane through the point $(1, 1, 1)$ and parallel to $\mathcal{P}_1$. What is the distance between the two planes?

3. Find the distance from the point $(2, 3, 2)$ to the ellipsoid $x^2 + 2y^2 + z^2 = 4$.

4. Find the minimal volume of a rectangular box with sides of length a, b, c, provided $\frac{1}{a^2} + \frac{4}{b^2} + \frac{9}{c^2} \leqslant 3$. The same question concerning the rectangular box of minimal surface.

5. For real numbers a, b consider the function $f(x, y, z) = \begin{cases} 4\frac{1}{x}e^{x^2y-5y^2z}\sin x + z & \text{if } x \neq 0 \\ ae^{-5y^2z} + bz & \text{if } x = 0 \end{cases}$

a) Show that if $a = 4, b = 1$, then $f(x, y, z)$ is continuous at all (x, y, z).

b) Find all the values of a, b for which $f(x, y, z)$ is continuous at all (x, y, z).

6. Find all the minimum, maximum, and saddle points of $f(x, y) = x^3 - y^3 + 6xy^2 - 150x$.

7) Let $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ be points in the disk of radius one centered at the origin, where $f(x, y) = x^2 + y^2 + 2y$ has an absolute maximum, respectively and absolute minimum. What is the distance from the origin to the line through P_0, P_1?

8) Let L be the normal line to the ellipsoid $2x^2 + 3y^2 + 10z^2 = 15$ at the point $(1, -1, 1)$. Find the points P on the y-axis which are closest to the line L.

9 Let $\vec{u}$ and $\vec{v}$ be unit vectors in the xyz-space.

a) Given that $|2\vec{u} - 3\vec{v}| = 2\sqrt{2}$, compute the magnitude of $\vec{u} + \vec{v}$.

b) If $|\vec{u} + \vec{v}| \neq \sqrt{2}$, then $\vec{u}$ and $\vec{v}$ are not orthogonal.

10. Let a function $h(x, y)$ have a saddle point at (x_0, y_0). Is it true that the Hessian determinant $\mathcal{H}_h(x_0, y_0)$ must satisfy $\mathcal{H}_h(x_0, y_0) < 0$? **Justify your answer!**

11. Consider the function $f(x, y) = \dfrac{\sin(x^3y^3)}{x^4 + y^4}$ for $(x, y) \neq (0,0)$ and $f(0,0) = a$.

a) Compute $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$, if the limit exists.

b) For which values of a is $f(x, y)$ continuous at all (x, y).

c) For a as above, is $f(x, y)$ differentiable at all (x, y)?

[**Hint**. $f(x, y) = \frac{x^3y^3}{x^4+y^4}\frac{\sin(x^3y^3)}{(x^3y^3)}$ and $x^2y^2 \leqslant 2x^2y^2 \leqslant x^4 + y^4$ (WHY), etc.]

12. Is it true that the function $f(x, y) = e^{\sin^2(xy)} + \ln(1 + x^4 + y^4)$ has an extremum at the point $(1, 1)$ in the region $-1 \leqslant x, y \leqslant 1$, despite $f_x(1, 1) \neq 0$? **Justify your answer!**

[**Hint**: How do the values of $\sin^2(xy)$, $1 + x^4 + y^4$ for $0 < x, y \leqslant 1$, $(x, y) \neq (1, 1)$ compare with $\sin^2(1)$, $1 + 1^4 + 1^4$, etc.]

13) Consider the ellipsoid $\mathcal{E}$: $x^2 + y^2 + 4z^2 = 4$, and the plane $\mathcal{P}$: $x + y + z = 6$.

a) Find the points P_0, P_1 on $\mathcal{E}$ which have minimal, respectively maximal, distance to $\mathcal{P}$.

b) What is the angle between the planes tangent to the ellipsoid $\mathcal{E}$ at P_0, respectively P_1?

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