Math 371 / Problem Set 2 (2 pages)

• Study thoroughly the Section 1, 2, 3, 4, 5 of the course notes Math XYZ

• Complete the proofs of Thm 3.1, Prop 3.4, Thm 3.5, Prop 3.8, Thm 3.9, Thm 3.15.

• Solve Exercises 3.13, 3.14.

• Complete the proofs from Thm 4.1, Remark 4.2, Prop 4.4, Thm 4.5.

1) A relation $R$ on a set $X$ is called a quasi-ordering, if $R$ is reflexive and transitive (but not necessarily anti-symmetric). For a quasi-ordering $R$ on $X$, define a relation $\sim R$ on $X$ by

\[
x \sim_R y \iff xRy \& yRx.
\]

Prove the following:

a) $\sim_R$ is an equivalence relation on $X$. Let $\hat{X} := X/\sim_R$ be the set of equivalence classes.

b) Define a relation $\leq$ on $\hat{X}$ by $\hat{x} \leq \hat{y} \iff xRy$. Then $\leq$ is an ordering on $\hat{X}$.

(*) What is the converse to a), b) above?

2) Let $X$ be a non-empty set, and recall the symmetric difference $A \Delta B := (A \setminus B) \cup (B \setminus A)$ on $\mathcal{P}(X)$. Prove/disprove the following:

a) The difference $A \setminus B$ on $\mathcal{P}(X)$ is not associative/commutative/has no neutral element.

b) $\mathcal{P}(X), \Delta$ is an abelian group.

3) In the notation from Problem 2) above, prove that $\mathcal{P}(X), \Delta, \cap$ is a commutative ring. Which elements in the ring $\mathcal{P}(X), \Delta, \cap$ are zero divisors, respectively invertible?

4) Prove/disprove the following:

a) Find the smallest $n_G$ such that $g^{n_G} = e_G$ for all $g \in G$, where i) $G = S_3$; ii) $G = S_7$.

b) A group $G$ is abelian iff $\forall x, y \in G$ one has: $x^2 y^2 = (xy)^2$.

5) Denote i) $A_1A_2A_3$ triangles; ii) $B_1B_2B_3B_4$ quadrangles; iii) $C_1C_2C_3C_4C_5$ pentagons. Depending of further properties of these shapes, write in each case the group of transformations as permutation groups of the vertices. [These groups will be subgroups of $S_3, S_4, S_5$ (WHY).]

The monoid/group/ring of functions

Let $X, T$ be non-empty sets, and $\text{Maps}(X,T) := \{f \mid f : X \to T \text{ abstract map}\}$. Given a composition law $\cdot$ on $T$, define the $\circ$ operation on $\text{Maps}(X,T)$ by $(f \circ g)(x) := f(x) \cdot g(x)$.

6) Prove/disprove the following assertions about the composition law $\circ$ defined above:

a) $\circ$ is associative, reps. commutative iff $\cdot$ is so.

b) $\circ$ has a neutral element $e$ iff $\cdot$ has a neutral element $e_\circ$. What is $e_\circ$ as a function?

c) $f \in \text{Maps}(X,T)$ is invertible w.r.t. $\circ$ iff $f(x) \in T$ is invertible w.r.t. $\cdot$ for all $x \in X$. 
7) In the context of Problem 4) above, prove or disprove the following:
   a) $G, \cdot$ is an (abelian) monoid, reps. group iff Maps$(X, G), \bullet$ is so.
   b) $R, +, \cdot$ is a (commutative) ring with $1_R$ iff the corresponding Maps$(X, R), \uparrow, \bullet$ is so.

   (●) Question: Is the same true for (skew) fields $R, +, \cdot$?

Language: Maps$(X, T)$ is called the monoid/group/ring of $T$-valued maps on $X$.

(Cartesian) products of algebraic structures

[The product of monoids/groups/rings/(skew) fields]

Let $\ast'$ and $\ast''$ be composition laws on $X'$, respectively $X''$. Define the coordinate wise composition law $\ast := \ast' \times \ast''$ on $X := X' \times X''$ by $(x', x'')(\ast'(y', y'')) := (x' \ast' y', x'' \ast'' y'')$.

8) Prove/disprove:
   a) $\ast$ is associative, reps. commutative if and only if $\ast'$ and $\ast''$ are so.
   b) $\ast$ has a neutral element $e$ iff $\ast'$ and $\ast''$ have neutral elements $e'$, $e''$.
   c) $x := (x', x'')$ is invertible iff $x'$ and $x''$ are invertible.

9) Let $G := G' \times G''$, $R := R' \times R''$ and $\ast = \ast' \times \ast''$, $\circ = \circ' \circ \circ''$. Prove the following:
   a) $G', \ast'$ and $G'', \ast''$ are (abelian) monoids, resp. groups, iff $G, \ast$ is so.
   b) $R', \ast', \circ'$ and $R'', \ast'', \circ''$ are (commutative) rings iff $R, \ast, \circ$ is so.

   (●) Question: Is the same true for fields $R', R''$?

Miscellaeia:

10) For a commutative ring $R, +, \cdot$ and $a, b, c, d \in R$, using $+$ and $\cdot$ define on $R$ a new “addition” by $x \oplus y = x + y + a$ and a new “multiplication” by $x \otimes y = xy + bx + cy + d$.
   a) Let $R = \mathbb{Z}$ or $R = \mathbb{R}$. Find all $a, b, c, d$ such that $\mathbb{Z}, \oplus, \otimes$ and/or $\mathbb{R}, \oplus, \otimes$ are rings.
   b) Solve in the ring $R, \oplus, \otimes$ the equations $x^2 \oplus 3x = 1_R$ and $x^2 \oplus 3\cdot x = 0_R$.

Generalized quaternions. Let $R$ be a commutative ring with $1_R \neq 0_R$, and define the (generalized) quaternions $\mathbb{H}_R$ over $R$ by: $\mathbb{H}_R := R^4 := \{a + b\imath + c\jmath + d\kappa \mid a, b, c, d \in R\}$ endowed with the coordinate wise addition $+$ and the multiplication $\cdot$ defined by:

$$i^2 = j^2 = k^2 = -1_R, \quad i \cdot j = k, \quad j \cdot k = i, \quad k \cdot i = j.$$ 

11) Prove that $\mathbb{H}_R$ is a ring with $1_{\mathbb{H}_R}$, and prove/disprove/answer the following:
   a)* $\mathbb{H}_Q$ and $\mathbb{H} := \mathbb{H}_R$ are skew fields. Solve the equation $(1 + \imath + j + \kappa) \cdot x = x \cdot \imath$ in $\mathbb{H}_Q, \mathbb{H}$.
   b)* $\mathbb{H}_R, +, \cdot$ is commutative iff $1_R = -1_R$ iff $\mathbb{H}_R \cong R^4, +, \cdot$ with component wise $+, \cdot$.
   c)* Let $\mathcal{M}_{2 \times 2}(R)$ be the ring of $2 \times 2$ matrices with $R$-coefficients. Prove/disprove:

$\mathbb{H}_R$ is isomorphic to $\mathcal{M}_{2 \times 2}(R)$ iff $\exists \imath \in R$ such that $\imath^2 = -1_R, \imath \neq -1_R$. 

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