Due: Friday, Sept 21, 2018

Math 371 / Problem Set 2 (2 pages)

- Study **thoroughly** the Section 1, 2, 3, 4, 5 of the course notes **Math XYZ**

- Complete the proofs of Thm 3.1, Prop 3.4, Thm 3.5, Prop 3.8, Thm 3.9, Thm 3.15.
- Solve Exercises 3.13, 3.14.
- Complete the proofs from Thm 4.1, Remark 4.2, Prop 4.4, Thm 4.5.

1) A relation R on a set X is called a quasi-ordering, if R is reflexive and transitive (but not necessarily anti-symmetric). For a quasi-ordering R on X, define a relation $\sim_R$ on X by

$$x \sim_R y \overset{\text{Def}}{\longleftrightarrow} xRy \;\&\; yRx.$$

Prove the following:

a) $\sim_R$ is an equivalence relation on X. Let $\widehat{X} := X/\sim_R$ be the set of equivalence classes.

b) Define a relation $\leqslant$ on $\widehat{X}$ by $\widehat{x} \leqslant \widehat{y} \overset{\text{Def}}{\longleftrightarrow} xRy$. Then $\leqslant$ is an ordering on $\widehat{X}$.

$(*)$ What is the converse to a), b) above ?

2) Let X be a non-empty set, and recall the symmetric difference $A \vartriangle B := (A \backslash B) \cup (B \backslash A)$ on $\mathcal{P}(X)$. Prove/disprove the following:

a) The difference $A \backslash B$ on $\mathcal{P}(X)$ is not associative/commutative/has no neutral element.

b) $\mathcal{P}(X), \vartriangle$ is an abelian group.

3) In the notation from Problem 2) above, prove that $\mathcal{P}(X), \vartriangle, \cap$ is a commutative ring. Which elements in the ring $\mathcal{P}(X), \vartriangle, \cap$ are zero divisors, respectively invertible?

4) Prove/disprove/answer the following:

a) Find the smallest n_G such that $g^{n_G} = e_G$ for all $g \in G$, where i) $G = S_3$; ii) $G = S_7$.

b) A group G is abelian iff $\forall\, x, y \in G$ one has: $x^2y^2 = (xy)^2$.

5) Denote i) $A_1A_2A_3$ triangles; ii) $B_1B_2B_3B_4$ quadrangles; iii) $C_1C_2C_3C_4C_5$ pentagons. Depending of further properties of these shapes, write in each case the group of transformations as permutation groups of the vertices. [These groups will be subgroups of S_3, S_4, S_5 (WHY).]

The monoid/group/ring of functions

Let X, T be non-empty sets, and $\text{Maps}(X, T) := \{f \mid f : X \to T \text{ abstract map}\}$. Given a composition law $\cdot$ on T, define the $\bullet$ operation on $\text{Maps}(X, T)$ by $(f \bullet g)(x) := f(x) \cdot g(x)$.

6) Prove/disprove the following assertions about the composition law $\bullet$ defined above:

a) $\bullet$ is associative, reps. commutative iff $\cdot$ is so.

b) $\bullet$ has a neutral element e iff $\cdot$ has a neutral element $e_\bullet$. What is $e_\bullet$ as a function?

c) $f \in \text{Maps}(X, T)$ is invertible w.r.t. $\bullet$ iff $f(x) \in T$ is invertible w.r.t. $\cdot$ for all $x \in X$.

7) In the context of Problem 4) above, prove or disprove the following:

a) $G, \cdot$ is an (abelian) monoid, reps. group iff $\mathrm{Maps}(X, G), \bullet$ is so.

b) $R, +, \cdot$ is a (commutative) ring with 1_R iff the corresponding $\mathrm{Maps}(X, R), \boxplus, \bullet$ is so.

($\bullet$) **Question:** Is the same true for (skew) fields $R, +, \cdot$?

Language: $\mathrm{Maps}(X, T)$ is called the monoid/group/ring of T-valued maps on X.

(Cartesian) products of algebraic structures

[The product of monoids/groups/rings/(skew) fields]

Let $*'$ and $*''$ be composition laws on X', respectively X''. Define the coordinate wise composition law $* := *' \times *''$ on $X := X' \times X''$ by $(x', x'') * (y', y'') := (x' *' y',\ x'' *'' y'')$.

8) Prove/disprove:

a) $*$ is associative, reps. commutative if and only if $*'$ and $*''$ are so.

b) $*$ has a neutral element e iff $*'$ and $*''$ have neutral elements e', e''.

c) $x := (x', x'')$ is invertible iff x' and x'' are invertible.

9) Let $G := G' \times G''$, $R := R' \times R''$ and $* = *' \times *''$, $\circ = \circ' \times \circ''$. Prove the following:

a) *$G', *'$ and $G'', *''$ are (abelian) monoids, resp. groups, iff $G, *$ is so.*

b) *$R', *', \circ'$ and $R'', *'', \circ''$ are (commutative) rings iff $R, *, \circ$ is so.*

($\bullet$) **Question:** Is the same true for fields R', R'' ?

Miscellaenia:

10) For a commutative ring $R, +, \cdot$ and $a, b, c, d \in R$, using $+$ and $\cdot$ define on R a new "addition" by $x \boxplus y = x + y + a$ and a new "multiplication" by $x \bullet y = xy + bx + cy + d$.

a) Let $R = \mathbb{Z}$ or $R = \mathbb{R}$. Find all a, b, c, d such that $\mathbb{Z}, \boxplus, \bullet$ and/or $\mathbb{R}, \boxplus, \bullet$ are rings.

b) Solve in the ring $R, \boxplus, \bullet$ the equations $x^2 \boxplus 3 \bullet x = 1_R$ and $x^2 \boxplus 3 \bullet x = 0_R$.

Generalized quaternions. Let R be a commutative ring with $1_R \neq 0_R$, and define the (generalized) quaternions $\mathbb{H}_R$ over R by: $\mathbb{H}_R := R^4 := \{a + b\imath + c\jmath + d\kappa \mid a, b, c, d \in R\}$ endowed with the coordinate wise addition $+$ and the multiplication $\cdot$ defined by:

$$\imath^2 = \jmath^2 = \kappa^2 = -1_R,\ \imath \cdot \jmath = \kappa,\ \jmath \cdot \kappa = \imath,\ \kappa \cdot \imath = \jmath.$$

11) Prove that $\mathbb{H}_R$ is a ring with $1_{\mathbb{H}_R}$, and prove/disprove/answer the following:

a)* $\mathbb{H}_{\mathbb{Q}}$ and $\mathbb{H} := \mathbb{H}_{\mathbb{R}}$ are skew fields. Solve the equation $(1 + \imath + \jmath + \kappa) \cdot x = x \cdot \imath$ in $\mathbb{H}_{\mathbb{Q}}, \mathbb{H}$.

b)* $\mathbb{H}_R, +, \cdot$ is commutative iff $1_R = -1_R$ iff $\mathbb{H}_R \cong R^4, +, \cdot$ with component wise $+, \cdot$.

c)* Let $\mathcal{M}_{2\times 2}(R)$ be the ring of 2×2 matrices with R-coefficients. Prove/disprove:

$$\mathbb{H}_R \textit{ is isomorphic to } \mathcal{M}_{2\times 2}(R) \textit{ iff } \exists\, \imath \in R \textit{ such that } \imath^2 = -1_R,\ \imath \neq -1_R.$$