Math 625 (Schemes & Curves) / Problem Set 2 (2 pages)

Schemes

1) Answer/prove/disprove the following:
   a) Describe all the closed (reduced) subschemes of \(\text{Spec } \mathbb{Z}, \text{Spec } k[t], \text{Spec } \mathbb{Z}[t], \text{Spec } k[t,u]\) where \(k\) is a field, and \(t, u\) are independent variables.
   b) \(\text{Spec}(R)\) is a quasi-compact for all commutative rings \(R\).
   c) For every scheme \(X\) there exists a unique scheme morphism \(X \rightarrow \text{Spec } \mathbb{Z}\).

2) Give detailed proofs of the assertions from the class:
   a) If \(X, \mathcal{O}_X\) is the affine scheme \(X = \text{Spec } R\), then \(R = \mathcal{O}_X(X)\) canonically.
   b) The category of commutative rings \(\text{Rings}\) is (anti)equivalent to the category of affine schemes \(\text{Sch}\) via the functor \(R \mapsto \text{Spec } R\) (HOW).
   c) Moreover, the category of commutative \(A\)-algebras \(A\)-\(\text{Alg}\) is (anti)equivalent to the category of \(A\)-schemes \(\text{Sch}_A\) (HOW).

3) Give detailed proofs of the assertions from the class:
   d) For every scheme \(X\) and every affine scheme \(Y = \text{Spec } R\), one has a canonical bijection:
      \[
      \text{Hom}_{\text{Rings}}(R, \mathcal{O}_X(X)) \leftrightarrow \text{Hom}_{\text{Sch}}(X, Y)
      \]
   b) The same question for \(A\)-algebras and \(A\)-schemes (HOW).

4) Let \(X = \bigcup \alpha \, \iota_\alpha(X_\alpha)\), and \(f : X \rightarrow Y\) be the result of gluing schemes \(X_\alpha, U_{\alpha\beta}, \phi_{\alpha\beta}, \ldots\) respectively scheme morphisms \(f_\alpha : X_\alpha \rightarrow Y_\alpha, \ldots\) in the category of ringed spaces. Prove:
   b) If \(X_\alpha = \bigcup i \in I_\alpha \, X_{\alpha i}\) is an affine open covering of each \(X_\alpha\), then \(\left(\iota_\alpha(X_{\alpha i})\right)_{\alpha,i}\) is an affine open covering of \(X\).
   b) The resulting \(f : X \rightarrow Y\) is a morphism of schemes.

Definition. A scheme \(X, \mathcal{O}_X\) is called reduced, if \(\mathcal{O}_X(U), U \subset X^{\text{top}}\) open, are reduced rings.

5) Prove the assertions from the class: For every scheme \(X\), TFAE:
   i) \(X\) is reduced.
   ii) There exists an affine open covering \(X = \bigcup \alpha X_\alpha, X_\alpha = \text{Spec } R_\alpha\) s.t. \(R_\alpha\) is reduced.
   iii) All the stalks \(\mathcal{O}_x\) of \(X\) are reduced local rings.

6) Let \(X\) be a scheme, and \(Y \hookrightarrow X\) be a closed subscheme of \(X\). Prove/disprove/answer:
   a) If \(X = \text{Spec } A\) is affine, there exists a reduced ideal \(a \subset X\) such that \(Y \hookrightarrow X\) factors uniquely as follows: \(\text{Spec } A/a \hookrightarrow Y \hookrightarrow X\). Moreover, \(Y^{\text{red}} = \text{Spec } A/a\) (WHY).
   b) Is there a corresponding assertion for arbitrary schemes \(X\)?

[Hint to b): Consider any affine open covering \(X = \bigcup Y_i\) of \(X\), and note that \(Y_i := Y \cap X_i\) satisfy: \(Y_i \subset Y\) are open, and endowed with \(\mathcal{O}_{Y_i}\) are closed immersions \(Y_i \hookrightarrow X_i\) (WHY), etc...]

7) Prove in all detail that products in \(\text{Sch} = \text{Sch}_\mathbb{Z}\) and \(\text{Sch}_T\) exist.

8) Let \(\varphi : T_1 \rightarrow T\) be separated in \(\text{Sch}\) and \(X, Y \in \text{Sch}_{T_1}\) be given. There exists a canonical \(T\)-morphism \(X \times_{T_1} Y \rightarrow X \times_T Y\) (HOW), which moreover, is a closed immersion.
Let \( R = \bigoplus_{d \geq 0} R_d \) be a graded comm. with \( 1_R \), and \( \text{Proj}(R) \subset \text{Spec}(R) \) be the proper homogeneous prime ideals of \( R \), i.e., not containing \( R_{>0} := \bigoplus_{d > 0} R_d \). For \( f \in R_d \), \( d > 0 \), set:

\[
R_{f,0} := \left\{ a/f^n \in R[1] \mid \deg(a) = \deg(f^n), \, n > 0 \right\}.
\]

Further, set \( D_f^+ := D_f \cap \text{Proj}(R) = \{ p \in \text{Proj}(R) \mid f \not\in p \} \subset \text{Spec}(R) \).

9) In the above context/notations, answer/prove/disprove the following:
   a) If \( a \subset R \) is homogeneous, so are \( \sqrt{a} = \mathcal{N}(a) \) and \( \mathcal{J}(a) \).
   Precisely, if \( p \in \text{Spec}(R) \) is a minimal with \( a \subset p \), then \( p \in \text{Proj}(R) \). What is the corresponding assertion for maximal ideals?
   b) Let \( n > 0 \) be fixed, and \( R^{(n)} := \bigoplus_{d \geq 0} R_{dn} \). Then \( \iota_n : R^{(n)} \hookrightarrow R \) is a morphism of \( R_0 \)-graded rings, and the resulting \( \text{Proj}(R) \to \text{Proj}(R^{(n)}) \) is a homeomorphism.
   c) \( \text{Proj}(R) \) is quasi-compact iff \( \exists d > 0 \exists f_1, \ldots, f_r \in R_d \) s.t. \( \sqrt{(f_1, \ldots, f_r)} = R_{>0} \).

10) Prove/disprove the assertions (some from the class):
   a) \( R_{f,0} \) is an \( R_0 \)-algebra, and the map: \( D_f^+ \to \text{Spec}(R_{f,0}), \, p \mapsto p_{f,0} := \cap R_{f,0} \) is a homeomorphism in the Zariski topology.
   b) Let \( \Sigma \subset R \) be a system of homogeneous elements in \( R_{>0} \). Then \( \bigcup_{f \in \Sigma} D_f^+ = \text{Proj}(R) \) iff the homogeneous ideal \( a_\Sigma \subset R \) generated by \( \Sigma \) satisfies: \( \sqrt{a_\Sigma} = R_{>0} \).

11) Recalling the sheaf of regular functions \( \mathcal{O}^{gr} \) on \( \text{Proj}(R) \), prove in all detail the assertion from the class (and in particular, that \( \text{Proj}(R) \) endowed with \( \mathcal{O}^{gr} \) is a scheme):

\[
D_f^+, \mathcal{O}^{gr}|_{D_f^+} \cong \text{Spec} R_{f,0} \text{ for every } f \in R_d, \, d > 0.
\]

**Definition.** Let \( R = \bigoplus_{d \geq 0} R_d, \, S = \bigoplus_{d \geq 0} S_d \) be graded rings with \( R_0, \, S_0 \) some \( A \)-algebras. Recall that \( R \otimes_A S := \bigoplus_{d \geq 0} R_d \otimes_A S_d \subset R \otimes_A S \) is the graded tensor product of the \( R \) and \( S \).

12) Prove/answer the following assertions (some from the class):
   a) Show that \( R \otimes_A S \) is a graded \( A \)-algebra.
   b) Is \( R \otimes_A S \) the coproduct of \( R \) and \( S \) in the category of graded \( A \)-algebras?
   c) \( \text{Proj}(R) \times_A \text{Proj}(S) = \text{Proj} R \otimes_A S \).

In particular, the class of \( \text{Proj} A \)-schemes is closed under products.

13) Proof in all detail the assertions from the class:
   a) The affine \( A \)-schemes are separated. Is the same true for general affine \( T \)-schemes?
   b) If \( X \in \text{Sch}_A \) satisfies: \( \forall x_1, x_2 \in X \exists U \subset \text{open affine s.t.} \, x_1, x_2 \in U \), then \( X \) is separated.
   c) The \( \text{Proj} A \)-schemes are separated. Is the same true for general \( \text{Proj} T \)-schemes?

**[Hint: Consider \( T \) as a \( Z \)-scheme and use Problem 8, etc. . . . ]**

14) Prove in all detail the assertions from the class:
   a) For \( f : X \to X', \, g : Y \to Y' \) in \( \text{Sch}_T \) \( \exists f \times g : X \times_T Y \to X' \times_T Y' \) canonically (HOW).
   b) If \( X_1 \hookrightarrow X, \, Y_1 \hookrightarrow Y \) are open/closed immersions in \( \text{Sch}_T \), so are \( X_1 \times_T Y_1 \to X \times_T Y \).

15) Prove/disprove the following assertions (some from the class):
   a) If \( Y \hookrightarrow X = \text{Spec} R \) is a closed immersion, then \( Y = \text{Spec} R/\mathfrak{a} \) for some ideal \( \mathfrak{a} \subset R \).
   b) Does the same hold correspondingly for closed immersions \( Y \hookrightarrow X = \text{Proj} R \)?
   **•** The same question, provided \( X = \text{Proj}(R) \) is quasi-compact.