Due: Spring 2019

Math 625 (Schemes & Curves) / Problem Set 2 (2 pages)

Schemes

1) Answer/prove/disprove the following:

a) Describe all the closed (reduced) subschemes of $\operatorname{Spec}\mathbb{Z}$, $\operatorname{Spec} k[t]$, $\operatorname{Spec}\mathbb{Z}[t]$, $\operatorname{Spec} k[t,u]$ where k is a field, and t, u are independent variables.

b) $\operatorname{Spec}(R)$ is a quasi-compact for all commutative rings R.

c) For every scheme X there exists a unique scheme morphism $X \to \operatorname{Spec}\mathbb{Z}$.

2) Give detailed proofs of the assertions from the class:

a) If $X, \mathcal{O}_X$ is the affine scheme $X = \operatorname{Spec} R$, then $R = \mathcal{O}_X(X)$ canonically.

b) The category of commutative rings **Rings** is (anti)equivalent to the category of affine schemes **Sch** via the functor $R \rightsquigarrow \operatorname{Spec} R$ (HOW).

c) Moreover, the category of commutative A-algebras A-**Alg** is (anti)equivalent to the category of A-schemes $\mathbf{Sch}_A$ (HOW).

3) Give detailed proofs of the assertions from the class:

d) For every scheme X and every affine scheme $Y = \operatorname{Spec} R$, one has a canonical bijection:

$$\operatorname{Hom}_{\mathbf{Rings}}\big(R, \mathcal{O}_X(X)\big) \longleftrightarrow \operatorname{Hom}_{\mathbf{Sch}}\big(X, Y\big)$$

b) The same question for A-algebras and A-schemes (HOW).

4) Let $X = \cup_\alpha \imath_\alpha(X_\alpha)$, and $f : X \to Y$ be the result of gluing schemes $X_\alpha, U_{\alpha\beta}, \phi_{\alpha\beta}, \ldots$ respectively scheme morphisms $f_\alpha : X_\alpha \to Y_\alpha, \ldots$ in the category of ringed spaces. Prove:

b) If $X_\alpha = \cup_{i \in I_\alpha} X_{\alpha i}$ is an affine open covering of each X_α, then $\big(\imath_\alpha(X_{\alpha i})\big)_{\alpha,i}$ is an affine open covering of X.

b) The resulting $f : X \to Y$ is a morphism of schemes.

Definition. A scheme $X, \mathcal{O}_X$ is called reduced, if $\mathcal{O}_X(U)$, $U \subset X^{\text{top}}$ open, are reduced rings.

5) Prove the assertions from the class: For every scheme X, TFAE:

i) X is reduced.

ii) There exists an affine open covering $X = \cup_\alpha X_\alpha$, $X_\alpha = \operatorname{Spec} R_\alpha$ s.t. R_α is reduced.

iii) All the stalks $\mathcal{O}_x$ of X are reduced local rings.

6) Let X be a scheme, and $Y \hookrightarrow X$ be a closed subscheme of X. Prove/disprove/answer:

a) If $X = \operatorname{Spec} A$ is affine, there exists a reduced ideal $\mathfrak{a} \subset X$ such that $Y \hookrightarrow X$ factors uniquely as follows: $\operatorname{Spec} A/\mathfrak{a} \hookrightarrow Y \hookrightarrow X$. Moreover, $Y^{\text{red}} = \operatorname{Spec} A/\mathfrak{a}$ (WHY).

b) Is there a corresponding assertion for arbitrary schemes X?

[**Hint** to b): Consider any affine open covering $X = \cup_i X_i$ of X, and note that $Y_i := Y \cap X_i$ satisfy: $Y_i \subset Y$ are open, and endowed with $\mathcal{O}_Y|_{Y_i}$ are closed immersions $Y_i \hookrightarrow X_i$ (WHY), etc...]

7) Prove in all detail that products in $\mathbf{Sch} = \mathbf{Sch}_{\mathbb{Z}}$ and $\mathbf{Sch}_T$ exist.

8) Let $\varphi : T_1 \to T$ be separated in **Sch** and $X, Y \in \mathbf{Sch}_{T_1}$ be given. There exists a canonical T-morphism $X \times_{T_1} Y \to X \times_T Y$ (HOW), which moreover, is a closed immersion.

Let $R = \oplus_{d\geq 0} R_d$ be a graded comm. with 1_R, and $\mathrm{Proj}(R) \subset \mathrm{Spec}(R)$ be the proper homogeneous prime ideals of R, i.e., not containing $R_{>0} := \oplus_{d>0} R_d$. For $f \in R_d$, $d > 0$, set:

$$R_{f,0} := \Big\{ a/f^n \in R[\tfrac{1}{f}] \;\Big|\; \deg(a) = \deg(f^n),\ n > 0 \Big\}.$$

Further, set $D_f^+ := D_f \cap \mathrm{Proj}(R) = \{\mathfrak{p} \in \mathrm{Proj}(R) \mid f \notin \mathfrak{p}\} \subseteq \mathrm{Spec}(R)$.

9) In the above context/notations, answer/prove/disprove the following:

a) If $\mathfrak{a} \subset R$ is homogeneous, so are $\sqrt{\mathfrak{a}} = \mathcal{N}(\mathfrak{a})$ and $\mathcal{J}(\mathfrak{a})$.
Precisely, if $\mathfrak{p} \in \mathrm{Spec}(R)$ is a minimal with $\mathfrak{a} \subset \mathfrak{p}$, then $\mathfrak{p} \in \mathrm{Proj}(R)$. What is the corresponding assertion for maximal ideals?

b) Let $n > 0$ be fixed, and $R^{(n)} := \oplus_{d\geqslant 0} R_{dn}$. Then $\imath_n : R^{(n)} \hookrightarrow R$ is a morphism of R_0-graded rings, and the resulting $\mathrm{Proj}(R) \to \mathrm{Proj}(R^{(n)})$ is a homeomorphism.

c) $\mathrm{Proj}(R)$ is quasi-compact iff $\exists\, d > 0\ \exists\, f_1, ..., f_r \in R_d$ s.t. $\sqrt{(f_1, ..., f_r)} = R_{>0}$.

10) Prove/disprove the assertions (some from the class):

a) $R_{f,0}$ is an R_0-algebra, and the map: $D_f^+ \to \mathrm{Spec}(R_{f,0})$, $\mathfrak{p} \mapsto \mathfrak{p}_{f,0} := \cap R_{f,0}$ is a homeomorphism in the Zariski topology.

b) Let $\Sigma \subset R$ be a system of homogeneous elements in $R_{>0}$. Then $\cup_{f\in\Sigma} D_f^+ = \mathrm{Proj}(R)$ iff the homogeneous ideal $\mathfrak{a}_\Sigma \subset R$ generated by Σ satisfies: $\sqrt{\mathfrak{a}_\Sigma} = R_{>0}$.

11) Recalling the sheaf of regular functions $\mathcal{O}^{\mathrm{gr}}$ on $\mathrm{Proj}(R)$, prove in all detail the assertion from the class (and in particular, that $\mathrm{Proj}(R)$ endowed with $\mathcal{O}^{\mathrm{gr}}$ is a scheme):

$$D_f^+, \mathcal{O}^{\mathrm{gr}}|_{D_f^+} \cong \mathrm{Spec}\, R_{f,0} \text{ for every } f \in R_d,\ d > 0.$$

Definition. Let $R = \oplus_{d\geq 0} R_d$, $S = \oplus_{d\geq 0} S_d$ be graded rings with R_0, S_0 some A-algebras. Recall that $R \otimes_A^{\mathrm{gr}} S := \oplus_{d\geq 0} R_d \otimes_A S_d \subset R \otimes_A S$ is the graded tensor product of the R and S.

12) Prove/answer the following assertions (some from the class):

a) Show that $R \otimes^{\mathrm{gr}} S$ is a graded A-algebra.

b) Is $R \otimes^{\mathrm{gr}} S$ the coproduct of R and S in the category of graded A-algebras?

c) $\mathrm{Proj}(R) \times_A \mathrm{Proj}(S) = \mathrm{Proj}\, R \otimes_A^{\mathrm{gr}} S$.
In particular, the class of Proj A-schemes is closed under products.

13) Proof in all detail the assertions from the class:

a) The affine A-schemes are separated. Is the same true for general affine T-schemes?

b) If $X \in \mathbf{Sch}_A$ satisfies: $\forall x_1, x_2 \in X\ \exists\, U \subset$ open affine s.t. $x_1, x_2 \in U$, then X is separated.

c) The Proj A-schemes are separated. Is the same true for general Proj T-schemes?

[**Hint**: Consider T as a $\mathbb{Z}$-scheme and use Problem 8, etc. . . .]

14) Prove in all detail the assertions from the class:

a) For $f : X \to X'$, $g : Y \to Y'$ in $\mathbf{Sch}_T$ $\exists\, f \times g : X \times_T Y \to X' \times_T Y'$ canonically (HOW).

b) If $X_1 \hookrightarrow X$, $Y_1 \hookrightarrow Y$ are open/closed immersions in $\mathbf{Sch}_T$, so are $X_1 \times_T Y_1 \to X \times_T Y$.

15) Prove/disprove the following assertions (some from the class):

a) If $Y \hookrightarrow X = \mathrm{Spec}\, R$ is a closed immersion, then $Y = \mathrm{Spec}\, R/\mathfrak{a}$ for some ideal $\mathfrak{a} \subset R$.

b) Does the same hold correspondingly for closed immersions $Y \hookrightarrow X = \mathrm{Proj}\, R$?

- The same question, provided $X = \mathrm{Proj}(R)$ is quasi-compact.