Math 625 (Schemes & Curves) / Problem Set 7 (two pages)

Basics (Differentials):
1) Let $B$ be an $A$-algebra. In the notations from the class, verify (in all detail) the following:
   a) Let $\Sigma \subset B$ be a multiplicative system. Then $\Omega^{\Sigma^{-1}/B|A} = \Sigma^{-1}\Omega^{1/B|A}$. In particular:
      $\Omega^{B|A} = (0)$ iff $\Omega^{B_q|A_p} = (0)$ for all $p \in \text{Spec}(A)$ iff $\Omega^{B_q|A} = (0)$ for all $q \in \text{Spec}(B)$
   b) Recall that $I := \ker (B \otimes_A B \to B)$, where $b' \otimes b'' \mapsto b'b''$. Then the canonical $B$-morphism $\Omega^{B|A} \to I/I^2$, $db \mapsto b \otimes 1 - 1 \otimes b$ is an isomorphism. (Make sure that you check all the details involving all the maps, e.g. well defined, etc.)

Let $A[X]$ be the polynomial ring in the variables $X := (X_\alpha)_{\alpha \in I}$, $a = (f) \subset A[X]$, where $f = (f_\beta)_\beta$, and $B := A[X]/a$. Hence $B = A[\underline{x}]$, where $\underline{x} := X \pmod{a}$.
2) Using the fundamental exact sequences, complete the proof the assertions form the class:
   a) $\Omega^{A[X]|A}$ is the free $A[X]$-module having $(dX_\alpha)_\alpha$ as a basis.
   b) $d\underline{x} := (dx_\alpha)_\alpha$ is a system of generators of $\Omega^{B|A}$, and $\Omega^{A[X]|A} \to \Omega^{B|A}$, $dX \mapsto d\underline{x}$, is a surjective morphism of $A[X]$-modules. [What is the $A[X]$-module structure of $\Omega^{B|A}$?]
   c) Moreover,
      
      \[
      \left( \sum_\alpha \frac{\partial f_\beta}{\partial X_\alpha} dX_\alpha \right)_\beta
      \]
      generates $\ker \left( \Omega^{A[X]|A} \to \Omega^{B|A} \right)$, hence $d\underline{x} := (dx_\alpha)_\alpha$ satisfy the relations $(\ast)$ above.

3) As applications, prove in all detail the assertions from the class:
   a) A finite $k$-algebra $R$ is separable iff $\Omega^{R|k} = (0)$.
   b) If $p = \text{char}(K) > 0$, and $K^p := \text{Frob}(F)$, then $(x_\alpha)_\alpha, x_\alpha \in K$, satisfies:
      $(x_\alpha)_\alpha$ is a $K^p$-basis of $K|kK^p$ iff $(dx_\alpha)_\alpha$ is a basis of $\Omega^{K|k}$.
   c) A transcendence basis $\mathcal{T} = (t_1, \ldots, t_d)$ of a function field $K|k$ is separable if and only if $d\mathcal{T} := (dt_1, \ldots, dt_d)$ is a $K$-basis of $\Omega^{K|k}$.

4) If a function field $K|k$ is separable generated, then $K|k$ is a formally smooth algebra.

Jacobian criterion, Tangent/cotangent space, Smooth points
Let $X$ be a $k$-prevariety. Recall the following facts explained in the class:

- For $X := (X_1, \ldots, X_n)$, $f = (f_1, \ldots, f_m)$ with $f_j \in k[X]$, let $\mathcal{J}_f := (\partial f_j/\partial X_i)_{j,i}$ be the Jacobian matrix of $f$. Further, for $U = \text{Spec} k[X]/(f)$, we let $\mathcal{J}_U$ be the image of $\mathcal{J}_f$ under $k[X] \to k[U]$, and by $\mathcal{J}_U(x)$ the image of $\mathcal{J}_U$ under $k[U] \to \kappa(x)$.

- A point $x \in X$ is called smooth, it the Jacobian criterion is satisfied at $x$, i.e., $\exists$ affine open neighborhoods $U = \text{Spec} k[X]/(f)$ of $x$ such that $\text{rk}(\mathcal{J}_U(x)) = n - \dim_x(X)$. 

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- If \( x \in X \) has \( \kappa(x)\vert k \) algebraic, \( T_{X,x} := \{ a \in \kappa(x)^m \mid a \cdot J_{U}(x)^\tau = 0 \} \subseteq \kappa(x)^m \) is the (classical) algebraic tangent space at \( x \in X \), and \( \kappa(x)^m/\{ a \cdot J_{U}(x)^\tau \mid a \in \kappa(x)^m \} \) is the (classical) cotangent space at \( x \in X \).

5) For \( x \in X \), complete the proofs of the assertions from the class:
   a) If \( x \in X \) is smooth, then the Jacobian criterion is satisfied at \( x \) for all affine open subsets \( x \in V = \text{Spec } k[\mathbf{Y}]/(g) \subseteq X \).
   b) Being a smooth point is invariant under base change, i.e., if \( k \rightarrow k' \) is a field extension, and \( p_X : Z = X \times_k k' \rightarrow X \) is the base change of \( X \) under \( k \rightarrow k' \), then \( Z_{\text{sm}} = p_X^{-1}(X_{\text{sm}}) \).
   c) The set of smooth points \( X_{\text{sm}} \subseteq X \) is open. In particular, \( X_{\text{sm}} \) is non-empty iff the function field \( k(X) = \prod_i k(X_i) \) has some \( k(X_i)\vert k \) separably generated.

6) Let \((X_i)_i\) be the irreducible components of \( X \), and suppose that \( x \in X \) has \( \kappa(x)\vert k \) algebraic. Prove in all detail the following assertions from the class:
   a) \( \Omega_{X/k}(x) \) and \( T_{X,x} = (m_x/m_x^2)^\lor \) are \( \kappa(x) \)-dual vector spaces, of dimension \( \geq \dim_x(X) \). And \( x \in X_{\text{sm}} \) if and only if \( \dim_{\kappa(x)} \Omega_{X/k}(x) = \dim_x(X) = \dim_{\kappa(x)} T_{X,x} \).
   b) If \( x \in X_{\text{sm}} \), then \( x \in X_{\text{reg}} \). And if \( x \in X_{\text{reg}} \) has \( \kappa(x)\vert k \) separable, then \( x \) is smooth.
   c) In particular, if \( x \in X_{\text{sm}} \), there exists a unique irreducible component \( X_\alpha \subseteq X \) such that \( x \in X_{\alpha,\text{sm}} \) and \( x \notin \cup_{\alpha' \neq \alpha} X_{\alpha'} \). Hence \( X_{\text{sm}} = \cup_\alpha (X_{\alpha,\text{sm}} \setminus X_\alpha) \), where \( X_\alpha := \cup_{\alpha' \neq \alpha} X_{\alpha'} \).
   (*) What are the corresponding assertions, if \( \kappa(x)\vert k \) is not necessarily algebraic?

[Hint: Use definitions and the second fundamental exact sequence, etc.]

7) Prove the following important fact:
   If \( x \in X_{\text{sm}}(k) \), then \( X \) is a local complete intersection at \( x \), i.e., there is an affine open neighborhood \( U = \text{Spec } k[\mathbf{X}]/(\mathbf{f}) \) of \( x \) such that \( n = d + m \), where \( d := \dim_x(X) \), and
   \[
   \left| \frac{\partial f_j}{\partial x_{i+d}} \right|_{1 \leq i, j \leq m} (x) \neq 0.
   \]
   Further, \( \mathbf{t} := (t_1, \ldots, t_d) \) with \( t_i := X_i \mod (\mathbf{f}) \) is a regular system of parameters at \( x \).

8) Compute the regular/smooth locus in the following cases:
   a) \( X = V(aX_1^3 - X_0X_2^2) \subseteq \mathbb{P}^2_k \), \( a \neq 0 \) (projective cuspidal cubic).
   b) \( X = V(2X_1^2 - 3X_2X_3) \subseteq \mathbb{A}^3_k \), \( Y = V(X_1X_2 - X_3) \subseteq \mathbb{A}^3_k \) and \( Z = X \cap Y \).
   c) \( X = V(aX_1^3 - bX_0X_2^2 - X_0X_3^2) \subseteq \mathbb{P}^2_k \), \( ab \neq 0 \) (projective nodal cubic)

The implicit Function Theorem. Let \( k \rightarrow \hat{k} \) be a field extension with \( \hat{k} \) a complete field w.r.t. an absolute value \( | \cdot | \), e.g., \( \hat{k} = \mathbb{C}, \mathbb{Q}_p \), etc. Then for every \( k \)-prevariety \( X \), the set \( X(\hat{k}) \) is a complete metric space in the strong topology \( \tau_{\vert_1} \). (Make sure you know what that is!)

9) In the above notation, let \( B_\epsilon(0) \subseteq \hat{k} \) be a ball of radius \( \epsilon \) centered at 0. For \( x \in X_{\text{sm}}(\hat{k}) \), let \( \mathbf{t} = (t_1, \ldots, t_d) \) be a system of regular parameters at \( x \). Show that there exist a \( \tau_{\vert_1} \)-open neighborhood \( U_x \subseteq X(\hat{k}) \) of \( x \), and \( \epsilon > 0 \) such that \( \mathbf{t} \) defines a \( \tau_{\vert_1} \)-diffeomorphism
   \[
   \mathbf{t} : U_x \rightarrow B_\epsilon(0)^d \subseteq \mathbb{A}^d(\hat{k})
   \]

[Hint: Use Problem 7 above and the Implicit Function Theorem, etc.]