Due: Spring 2019

Math 625 (Schemes & Curves) / Problem Set 7 (two pages)

Basics (Differentials):

1) Let B be an A-algebra. In the notations from the class, verify (in all detail) the following:

a) Let $\Sigma \subset B$ be a multilplicative system. Then $\Omega_{\Sigma^{-1}B|A} = \Sigma^{-1}\Omega_{B|A}$. In particular:

$$\Omega_{B|A} = (0) \;\text{ iff }\; \Omega_{B_{\mathfrak{p}}|A_{\mathfrak{p}}} = (0)\ \forall\ \mathfrak{p} \in \operatorname{Spec}(A) \;\text{ iff }\; \Omega_{B_{\mathfrak{q}}|A} = (0)\ \forall\ \mathfrak{q} \in \operatorname{Spec}(B)$$

b) Recall that $I := \ker\big(B \otimes_A B \to B\big)$, where $b' \otimes b'' \mapsto b'b''$. Then the canonical B-morphism $\Omega_{B|A} \to I/I^2$, $db \mapsto b \otimes 1 - 1 \otimes b$ is an isomorphism. (Make sure that you check all the details involving all the maps, e.g. well defined, etc.)

Let $A[\underline{\boldsymbol{X}}]$ be the polynomial ring in the variables $\underline{\boldsymbol{X}} := (X_\alpha)_{\alpha \in I}$, $\mathfrak{a} = (\underline{\boldsymbol{f}}) \subset A[\underline{\boldsymbol{X}}]$, where $\underline{\boldsymbol{f}} = (f_\beta)_\beta$, and $B := A[\underline{\boldsymbol{X}}]/\mathfrak{a}$. Hence $B = A[\underline{\boldsymbol{x}}]$, where $\underline{\boldsymbol{x}} := \underline{\boldsymbol{X}} \pmod{\mathfrak{a}}$.

2) Using the fundamental exact sequences, complete the proof the assertions form the class:

a) $\Omega_{A[\underline{\boldsymbol{X}}]\,|\,A}$ is the free $A[\underline{\boldsymbol{X}}]$-module having $(dX_\alpha)_\alpha$ as a basis.

b) $d\underline{\boldsymbol{x}} := (dx_\alpha)_\alpha$ is a system of generators of $\Omega_{B|A}$, and $\Omega_{A[\underline{\boldsymbol{X}}]\,|\,A} \to \Omega_{B|A}$, $d\underline{\boldsymbol{X}} \mapsto d\underline{\boldsymbol{x}}$, is a surjective morphism of $A[\underline{\boldsymbol{X}}]$-modules. [What is the $A[\underline{\boldsymbol{X}}]$-module structure of $\Omega_{B|A}$?]

c) Moreover,

$$(*) \qquad \Big(\sum_\alpha \frac{\partial f_\beta}{\partial X_\alpha}\, dX_\alpha\Big)_\beta$$

generates $\ker\Big(\Omega_{A[\underline{\boldsymbol{X}}]\,|\,A} \twoheadrightarrow \Omega_{B|A}\Big)$, hence $d\underline{\boldsymbol{x}} := (dx_\alpha)_\alpha$ satisfy the relations $(*)$ above.

3) As applications, prove in all detail the assertions from the class:

a) A finite k-algebra R is separable iff $\Omega_{R|k} = (0)$.

b) If $p = \operatorname{char}(K) > 0$, and $K^p := \operatorname{Frob}(F)$, then $(x_\alpha)_\alpha$, $x_\alpha \in K$, satisfies:
$(x_\alpha)_\alpha$ is a K^p-basis of $K \,|\, kK^p$ iff $(dx_\alpha)_\alpha$ is a basis of $\Omega_{K|k}$.

c) A transcendence basis $\mathcal{T} = (t_1, \ldots, t_d)$ of a function field $K|k$ is separable if and only if $d\mathcal{T} := (dt_1, \ldots, dt_d)$ is a K-basis of $\Omega_{K|k}$.

4) If a function field $K|k$ is separable generated, then $K|k$ is a formally smooth algebra.

Jacobian criterion, Tangent/cotangent space, Smooth points

Let X be a k-prevariety. Recall the following facts explained in the class:

- For $\underline{\boldsymbol{X}} = (X_1, \ldots, X_n)$, $\underline{\boldsymbol{f}} = (f_1, \ldots, f_m)$ with $f_j \in k[\underline{\boldsymbol{X}}]$, let $\mathcal{J}_{\underline{\boldsymbol{f}}} := \big(\partial f_j/\partial X_i\big)_{j,i}$ be the Jacobian matrix of $\underline{\boldsymbol{f}}$. Further, for $U = \operatorname{Spec} k[\underline{\boldsymbol{X}}]/(\underline{\boldsymbol{f}})$, we let $\mathcal{J}_U$ be the image of $\mathcal{J}_{\underline{\boldsymbol{f}}}$ under $k[\underline{\boldsymbol{X}}] \twoheadrightarrow k[U]$, and by $\mathcal{J}_U(x)$ the image of $\mathcal{J}_U$ under $k[U] \twoheadrightarrow \kappa(x)$.
- A point $x \in X$ is called smooth, it the Jacobian criterion is satisfied at x, i.e., $\exists$ affine open neighborhoods $U = \operatorname{Spec} k[\underline{\boldsymbol{X}}]/(\underline{\boldsymbol{f}})$ of x such that $\operatorname{rk}\Big(\mathcal{J}_U(x)\Big) = n - \dim_x(X)$.

- If $x \in X$ has $\kappa(x)|k$ algebraic, $T_{X,x} := \{\underline{\boldsymbol{a}} \in \kappa(x)^n \,|\, \underline{\boldsymbol{a}} \cdot \mathcal{J}_U(x)^\tau = 0\} \subset \kappa(x)^n$ is the (classical) tangent space at $x \in X$, and $\kappa(x)^n/\{\underline{\boldsymbol{a}} \cdot \mathcal{J}_U(x)^\tau \,|\, \underline{\boldsymbol{a}} \in \kappa(x)^n\}$ is the (classical) cotangent space at $x \in X$.

5) For $x \in X$, complete the proofs of the assertions from the class:

a) If $x \in X$ is smooth, then the Jacobian criterion is satisfied at x for all affine open subsets $x \in V = \operatorname{Spec} k[\underline{\boldsymbol{Y}}]/(\underline{\boldsymbol{g}}) \subset X$.

b) Being a smooth point is invariant under base change, i.e., if $k \hookrightarrow k'$ is a field extension, and $p_X : Z = X \times_k k' \to X$ is the base change of X under $k \hookrightarrow k'$, then $Z_{\mathrm{sm}} = p_X^{-1}(X_{\mathrm{sm}})$.

c) The set of smooth points $X_{\mathrm{sm}} \subset X$ is open. In particular, X_{sm} is non-empty iff the function field $k(X) = \prod_i k(X_i)$ has some $k(X_i)|k$ separably generated.

6) Let $(X_i)_i$ be the irreducible components of X, and suppose that $x \in X$ has $\kappa(x)|k$ algebraic. Prove in all detail the following assertions from the class:

a) $\Omega_{X|k}(x)$ and $T_{X,x} = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ are $\kappa(x)$-dual vector spaces, of dimension $\geqslant \dim_x(X)$. And $x \in X_{\mathrm{sm}}$ if and only if $\dim_{\kappa(x)} \Omega_{X|x}(x) = \dim_x(X) = \dim_{\kappa(x)} T_{X,x}$.

b) If $x \in X_{\mathrm{sm}}$, then $x \in X_{\mathrm{reg}}$. And if $x \in X_{\mathrm{reg}}$ has $\kappa(x)|k$ separable, then x is smooth.

c) In particular, if $x \in X_{\mathrm{sm}}$, there exists a *unique* irreducible component $X_\alpha \subset X$ such that $x \in X_{\alpha,\mathrm{sm}}$ and $x \notin \cup_{\alpha' \neq \alpha} X_{\alpha'}$. Hence $X_{\mathrm{sm}} = \cup_\alpha (X_{\alpha,\mathrm{sm}} \backslash X'_\alpha)$, where $X'_\alpha := \cup_{\alpha' \neq \alpha} X_{\alpha'}$.

$(*)$ What are the corresponding assertions, if $\kappa(x)|k$ is not necessarily algebraic?

[**Hint:** Use definitions and the second fundamental exact sequence, etc.]

7) Prove the following important fact:

If $x \in X_{\mathrm{sm}}$, then X is a local complete intersection *at x, i.e., there is an affine open neighborhood $U = \operatorname{Spec} k[\underline{\boldsymbol{X}}]/(\underline{\boldsymbol{f}})$ of x such that $n = d + m$, where $d := \dim_x(X)$, and*

$$\left|\mathcal{J}_{\underline{\boldsymbol{f}}}(x')\right| = \left|\frac{\partial f_j}{\partial X_{i+d}}\right|_{1 \leqslant i,j \leqslant m}(x') \neq 0 \quad \forall\, x' \in U.$$

Further, $(dt_1, \ldots, dt_d)$ with $t_i := X_i \left(\operatorname{mod} (\underline{\boldsymbol{f}})\right)$ is an $\mathcal{O}_{X,x'}$ basis of $\Omega_{X|k,x'}$ for all $x' \in U$.

8) Compute the regular/smooth locus in the following cases:

a) $X = V(aX_1^3 - X_0X_2^2) \subset \mathbb{P}^2_k$, $a \neq 0$ (projective cuspidal cubic).

b) $X = V(2X_1^2 - 3X_2X_3) \subset \mathbb{A}^3_k$, $Y = V(X_1X_2 - X_3) \subset \mathbb{A}^3_k$ and $Z = X \cap Y$.

c) $X = V(aX_1^3 - bX_0X_1^2 - X_0X_2^2) \subset \mathbb{P}^2_k$, $ab \neq 0$ (projective nodal cubic)

The implicit Function Theorem. Let $k \hookrightarrow \hat{k}$ be a field extension with $\hat{k}$ a complete field w.r.t. an absolute value $|\ |$, e.g., $\hat{k} = \mathbb{C}, \mathbb{Q}_p$, etc. Then for every k-prevariety X, the set $X(\hat{k})$ is a complete metric space in the strong topology $\tau_{|\,|}$. (Make sure you know what that is!)

9) In the notation from Problem 7), for $\hat{x}' \in U(\hat{k})$, say defined by $x' \in U$ and $\kappa(x') \hookrightarrow \hat{k}$, set $\hat{a}' := \underline{\boldsymbol{t}}(\hat{x}') \in \hat{k}^d$, and for $\epsilon > 0$ and $\hat{a} \in \hat{k}^d = \mathbb{A}^d(\hat{k})$, set $B_\epsilon(\hat{a}) := \{\hat{a}' \in \hat{k}^d \,\big|\, \|\hat{a}' - \hat{a}\| < \epsilon\}$. (Make sure you know/understand all the notations!) Prove:

For all $\hat{x} \in U(\hat{k})$ there is a $\tau_{|\,|}$-open neighborhood $\mathcal{U}_{\hat{x}} \subset X(\hat{k})$ of $\hat{x}$ and $\epsilon > 0$ such that

$$\underline{\boldsymbol{t}} : \mathcal{U}_x \to B_\epsilon(\hat{a}), \quad \hat{x}' \mapsto \hat{a}' := \underline{\boldsymbol{t}}(\hat{x}') \quad \textit{is a homeomorphism.}$$

[**Hint:** Use Problem 7 above and the Implicit Function Theorem, etc.]