Math 202 / Midterm 1 (two pages)

Academic Integrity Statement:
My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this Math 202 exam. [That means, among other things, that you are allowed to: (a) discuss the problems of the midterm with your colleagues, but not work out solutions together; (b) ask any member of the Math Dept about hints to the exam problems, but you must first mention to her/him that it goes about problems on a take home exam; (c) consult resources (books, Internet, etc.), but you must design/write down your own proofs.]

Name (printed): __________________________ Signature:

Note: There are 8 (eight) problems on this exam.
Points: 12.5 points for each problem (extra and/or partial credit possible).
Grading: 50 < C−, C, C+ < 70 < B−, B, B+ < 85 < A−, A, A+
Procedures: Print out the two pages of the exam and staple them to your work. Write your name (printed) and sign the above Academic Integrity Statement.

Recall: A complete proof must contain all the necessary explanations/steps, and in order to disprove an assertion you must give a counterexample showing that the assertion is not true.

1) Consider the assertion in plain English:
“Every natural number less than 1024 is a sum of four squares of natural numbers.”
a) Write the above assertion using quantifiers.
b) What is the negation of the above assertion in plain English.
c) Write the negation of the above assertion using quantifiers.
d) Is the above assertion true?

2) Prove/disprove/answer the following:
a) If \( A, B, C, D, F \) are sets, then there exists a set \( X \) whose elements are precisely the sets \( A \cup B, C \setminus D, F \times F, A \cup F, A \cup C \). Is \( A \) an element of \( X \)?
b) Let \( f : A \to B, g : B \to A \) be maps. If \( g \circ f \) is injective, respectively surjective, so are \( f \) and \( g \). If \( g \circ f \) and \( f \circ g \) are injective, respectively surjective, so are \( f \) and \( g \).

3) Let \( A, \leq \) and \( A', \leq' \) be totally ordered sets, and \( f : A \to A' \) be strictly increasing, i.e., \( \forall x_1, x_2 \in A \) one has: \( x_1 < x_2 \Rightarrow f(x_1) <' f(x_2) \). Prove/disprove/answer the following:
a) If \( A \) has 19 elements, and \( f \) is strictly increasing, then \( A' \) has more than 5 elements. What is the converse of this assertion, and is the converse true? (Justify your answer!)
b) Suppose that $|A| = 3$ and $|A'| = n$ is a finite set. How many increasing, respectively strictly increasing, maps $f : A \to A'$ are there? (Justify your answer!)

4) Prove/disprove/answer the following:
   a) For any two total orderings $\le, \le'$ on a set $A$ of cardinality $|A| = 2019$ there is a unique bijection $f : A \to A$ satisfying: $x < y$ if and only if $f(x) < f(y)$, for all $x, y \in A$.
   b) Is the same true if $A$ is infinite?

5) Recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for all $n, k \in \mathbb{N}$, $n \geq k$, where $0! \overset{\text{def}}{=} 1$. Prove the following:
   a) $\binom{n}{k} = \binom{n}{n-k}$ for $0 \leq k \leq n$, and $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for $0 < k \leq n - 1$.
   b) How many equivalence relations are there on a set with 10 elements?

6) Define on $\mathbb{N}$ the relation $\sim$ by:
   $$m \sim n \overset{\text{def}}{\iff} m \text{ and } n \text{ give the same remainder under division by 99}.$$  
   Prove/disprove/answer the following:
   a) $\sim$ is an equivalence relation on $\mathbb{N}$.
   b) What are the equivalence classes of 0, 1, 101?

7) Let $R$ denote either $\mathbb{Z}, +, \cdot$ or $\mathbb{Q}, +, \cdot$, and for $a, b, c \in R$ (using the usual $+$ and $\cdot$) define on $R$ a new addition by $x \oplus y = x + y + 1$ and a new multiplication by $x \odot y = xy + ax + by + c$.
   a) Find all $a, b, c \in R$ such that $R$ endowed with $\oplus, \odot$ is a ring (what does that mean?).
   b) Solve in the ring $R$ the equations $x \odot x \oplus 3 \odot x = 1_R$ and $x \odot x \oplus 3 \odot x = 0_R$.

8) Recall that $\le$ is defined in $\mathbb{Z}$ by $a \le b \overset{\text{def}}{\iff} \exists k \in \mathbb{N}$ s.t. $a + k = b$. Answer the following:
   a) The set $A := \{x \in \mathbb{Z} \mid x \geq -3\}$ is well ordered w.r.t. the ordering $\le$.
   b) For all $a, b \in \mathbb{Z}$ one has: $a \cdot b > 0_\mathbb{Z}$ iff either $a, b > 0_\mathbb{Z}$ or $a, b < 0_\mathbb{Z}$.