Math 202 / Problem Set 2
(two pages)

Due: Fr, Sept 20, 2019 (in class)

Basics of logical deduction

1) Consider the assertions in plain English: $p \equiv$ (all doors will open), $q \equiv$ (the train stops). Answer the following:
   a) What is the (logical) negation in plain English of the assertion $p$, respectively $q$.
   b) Using $\neg$, $\&$, $\lor$ and $p$, $q$, write down the following assertion:
      “Either all doors will open, or the train does not stop.”

2) Let $p, q$ be the assertions $p \equiv$ (more jobs), $q \equiv$ (lower taxes), $r \equiv$ (increase spending). In country $X$, statistical data show that lately more jobs were created. Explain what is logically faulty with the following assertions —dear to some politicians/economists/others:
   a) “You see, since we lowered taxes, more jobs were created.”
   b) “You see, because we did not lower the taxes, we could increase spending, and therefore more jobs were created.”

   [Hint: Write down the assertion a), b) as logical assertions using $p, q, r$, and see whether the cause-effect is the one explained by politicians/economists/some; recall that “wrong implies everything”…]

Sets and Functions

3) Let $A, B, C, D$ be given sets, and $x$ be elements, e.g., real numbers. Answer the following:
   a) Using $\cup$, $\cap$, $\setminus$ and $A, B, C, D$ write down the sets of all $x$ which satisfy:
      i) $(x \in A \text{ or } x \in B) \& x \in C \& x \notin D$;
      ii) $x \in A \text{ or } (x \in B \& x \in C) \& x \notin D$.
   b) Write as a union of disjoint intervals the sets of the real numbers $x \in \mathbb{R}$ satisfying:
      i) $(x < 20 \& x^2 < 100) \text{ or } x \notin (-\infty, -1]$;
      ii) $x < 20 \& (x^2 < 100 \text{ or } x \notin (-\infty, -1])$.
   (*): Does the place of the parentheses matter?

4) Let $A$, $B$ be sets. Answer the following:
   a) $\exists f: A \to B$ injective iff $\exists g: B \to A$ surjective.
   b) $f: A \to B$ is injective iff $\exists g: B \to A$ surjective satisfying $g(f(x)) = x$ for all $x \in A$.
   c) $f: A \to B$ is surjective iff $\exists g: B \to A$ injective satisfying $f(g(y)) = y$ for all $y \in B$.

Cardinality of sets. Recall that the cardinality of a set $A$, denoted by $|A|$, is, intuitively, a kind of size of $A$. Recall that $|A| \leq |B|$ $\iff$ $\exists f: A \to B$ injective, and that one has:

Theorem (Cantor, Bernstein, Schroeder). $|A| \leq |B|$ and $|B| \leq |A|$ if and only if there exists a function $f: A \to B$ which is bijective.

- Recall the definitions from the class (see Notes): $[0] = \emptyset$, $[n] = \{1, \ldots, n\}$ for $n \neq 0$, and:
  - $X$ is called finite of cardinality $|X| = n \geq 0$, if $\exists f: [n] \to X$ bijective.
    If so, and $n \neq 0$, then $X = \{x_1, \ldots, x_n\}$, where $x_i = f(i)$, $1 \leq i \leq n$.
  - $X$ is called countable, if $|X| = |\mathbb{N}|$, i.e., there exists a bijection $f: X \to \mathbb{N}$.
  - $X$ is called at most countable, if $|X| \leq |\mathbb{N}|$. 

1
5) Let $X$ be a non-empty set. Prove/disprove the following:
   a) If $X$ is finite, then every injective (resp. surjective) map $f : X \to X$ is bijective.
   b) If $X$ is infinite, there exists injective (surjective) $f : X \to X$ which are not bijective.

6) Let $X$ be an arbitrary set, and $\mathcal{P}(X) := \{A \mid A \subseteq X\}$ be the power set of $X$. Prove:
   a) If $|X| = n$ is finite, then $|\mathcal{P}(X)| = 2^n$.
   b) One has always: $|X| < |\mathcal{P}(X)|$ (Google it!). Deduce from this that $|\mathbb{N}| < |\mathbb{R}|$.

[Hint to the second part of b): Define $f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ by $f(A) := a_0 \cdot a_1 a_2 \cdots a_n \cdots$ for $A \subseteq \mathbb{N}$, where $a_n = 1$ if $n \in A$, and $a_n = 0$ if $n \notin A$. Then $A \neq A'$ implies $x_A \neq x_{A'}$ (WHY), hence $f$ is injective, etc...]

7) Let $X, Y$ be finite sets, say $|X| = m$ and $|Y| = n$. Prove/disprove the following assertions:
   a) $|X \cup Y| + |X \cap Y| = |X| + |Y|$. What is the corresponding assertion for $|X \cup Y \cup Z|$?
   b) $|X \times Y| = |X| \cdot |Y|$. What is the corresponding assertion for $|X \times Y \times Z|$?

8) Let $A, B, A_n, n \in \mathbb{N}$ be at most countable sets. Prove the following assertions:
   a) $A \times B$ is at most countable. Is the same true for $A_0 \times \cdots \times A_n$ for all $n \in \mathbb{N}$.
   b) $A \cup B$ is at most countable. Is the same true for $\cup_{n \in \mathbb{N}} A_n$?
   c) Is the same true for the (infinite cartesian) product $A_0 \times \cdots \times A_n \times \ldots$?

More about the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$

9) Complete the proof of the assertions from the class:
   a) $+$ and $\cdot$ in $\mathbb{N}$ are associative, commutative, and 0, respectively 1 are neutral elements. Further, $\cdot$ is distributive w.r.t to $+$
   b) $+$ and $\cdot$ have cancelation property in $\mathbb{N}$, respectively $\mathbb{N}_{>0}$, i.e., for $n, m, k \in \mathbb{N}$ one has:
   - $m + k = n + k \Rightarrow m = n$.
   - $m \cdot k = n \cdot k \Rightarrow m = n$, provided $k > 0$.

10) Recall that for $m, n \in \mathbb{N}$, one says that $m \leq n$, if there exists $k \in \mathbb{N}$ such that $n = m + k$.
    Complete the proof of the assertions from the class:
    \[ \leq \text{ is compatible with } + \text{ and } \cdot \]
    That is, for all $n, m, k \in \mathbb{N}$ one has:
    a) $m \leq n$ iff $m + k \leq n + k$.
    b) $m \leq n$ iff $m \cdot k \leq n \cdot k$, provided $k > 0$. 

2