Math 620 (Algebraic Number Theory I), PS 1 (two pages)

Integral ring extensions

1) Let $A$ be an integrally closed domain, $K = Quot(A)$, and for $L|K$ algebraic field extension, let $B \subset L$ the integral closure of $A$ in $L$. For $p \in \text{Spec}(A)$, let $X_p \subset \text{Spec}(B)$ be the fiber of $\text{Spec}(B) \to \text{Spec}(A)$. Prove/disprove the following:
   a) For $x \in L$, let $p_x(T) \in K[T]$ be its minimal polynomial. Then $x \in B$ iff $p_x(T) \in A[T]$.
   b) Let $x \in L$ is integral over $A$ iff $x$ is integral over $A_m$ for all $p \in \text{Max}(A)$.
   c) Let $L|K$ be finite separable. Then $A$ is Noetherian iff $B$ is Noetherian.
   d) Any distinct $q, q' \in X_p$ are incomparable w.r.t. inclusion, and $|X_p| \leq |L : K|$.

Valuations

2) Let $\mathcal{O} \subset K$ be a valuation ring of $K$. Prove/disprove:
   a) Every finitely generated $\mathcal{O}$-submodule $M$ of $K$, + is principal.
   b) Every ideal of $\mathcal{O}$ is principal.
   c) Any two $\mathcal{O}$-submodules of $K$, + are comparable w.r.t. inclusion.

3) Let $\mathcal{M}, \leq$ be the lattice of all $\mathcal{O}$-submodules of $K$, +, and $\leq$ be the ordering defined by inclusion. Then there is a canonical embedding $v_0 K \hookrightarrow \mathcal{M}$ compatible with the ordering.

4) Prove the assertion from the class: Let $\mathcal{O}_{v_1} \subset K$ be a valuation ring with valuation ideal $m_{v_1}$, and $\mathcal{O}_{v_0}$ be a valuation ring of $K_0 := \kappa(v_1)$ with valuation ideal $m_{v_0}$ and residue field $\kappa(v_0)$. Let $\mathcal{O}, m \subset K$ be the preimages of $\mathcal{O}_{v_0}, m_{v_0}$ under $\mathcal{O}_{v_1} \to \kappa(v_1) = K_0$. Prove:
   a) $\mathcal{O} \subset K$ is a valuation ring, saay $\mathcal{O} = \mathcal{O}_v$, with valuation ideal $m_v$ and $\kappa(v) = \kappa(v_0)$.
   b) There is a canonical exact sequence of totally ordered group $0 \to \Gamma_{v_0} \to \Gamma_v \to \Gamma_{v_1} \to 0$.

Notation/Language. $v$ is called the (valuation theoretical) composition of $v_0$ and $v_1$, and $v_0$ is called the quotient of $v$ by $v_1$. Notations: $v = v_0 \circ v_1$, $v_0 = v/v_1$.

5) Let $\mathcal{O} \subset K$ be a valuation ring, and define: $\mathcal{R} := \{ \mathcal{O}' \mid \mathcal{O} \subset \mathcal{O}' \subset K \text{ subring} \}$ and $\mathcal{G} := \{ \Delta \mid \Delta \leq \kappa_{\mathcal{O}} \text{ convex subgroup} \}$. Prove the assertions from the class:
   a) Every $\mathcal{O}' \in \mathcal{R}$ is a valuation ring, and its valuation ideal $m'$ satisfies: $m' = m' \cap \mathcal{O}$, and $\mathcal{O}' = \mathcal{O}_{m'}$. Conversely, if $p \in \text{Spec}(\mathcal{O})$, then $\mathcal{O}' := \mathcal{O}_p \in \mathcal{R}$, and $m' = p_p = p$.
   b) Every $\Gamma_\Delta := \Gamma_{\mathcal{O}}/\Delta$ is a totally ordered group, and $\kappa : K \xrightarrow{\text{can}} \Gamma_{\mathcal{O}} \xrightarrow{\Delta} \Gamma_{\mathcal{O}}'$ defines a valuation with $\mathcal{O}' := \mathcal{O}_{\kappa'} \in \mathcal{R}$, and $\Delta = \text{ker}(\Gamma_{\mathcal{O}} \to \Gamma_{\mathcal{O}'})$.
   c) The above recipes define conical bijections (explain how): $\mathcal{R} \leftrightarrow \text{Spec}(\mathcal{O}) \leftrightarrow \mathcal{G}$

Absolute Values

6) Prove that for absolute values $| |, | |'$ on $K$ the following are equivalent:
   i) The (metric) topologies $\tau_{| |}, \tau_{| |'}$ are identical.
   ii) There is a real number $\rho > 0$ such that $| |' = | |^\rho$.
   i) One has equality of the open balls of radius one ($|x| < 1 = (|x|' < 1)$.

Recall that if $| |, | |'$ satisfy i), ii), iii), they are called equivalent. Notation $| | \sim | |'$.
• Study the proofs of (or give your own proofs to) Ostrowski’s Theorems:

**Thm A.** Let \(|\cdot|'\) be a non-trivial value of \(K\) such that \(\exists n \in \mathbb{N} \text{ with } |n \cdot 1_K'| > 1\). Then there exists a unique absolute value \(|\cdot|_K \sim |\cdot|'\) and a unique isometric embedding
\[ K, |\cdot|_K \hookrightarrow \mathbb{C}, |\cdot|. \]

**Thm B.** Every absolute value of \(\mathbb{Q}\) is equivalent to either the usual archimedean absolute value \(|\cdot|\) or to a unique canonical absolute value \(|\cdot|_p\).

Recall that an absolute value of \(K\) satisfying \(|n \cdot 1_K'| \leq 1\) for all \(n \in \mathbb{N}\) is called non-archimedean. Hence every absolute value is either archimedean, or non-archimedean.

7) Prove the assertions from the class:
   a) A non-archimedean absolute value \(|\cdot|\) satisfies the ultrametric triangle inequality:
   \[ |x + y| \leq \max(|x|, |y|), \text{ and } |x + y| = \max(|x|, |y|) \text{ if } |x| \neq |y|. \]
   What is the corresponding assertion for general valuations?
   b) The set of equivalence classes of absolute values of \(K\) is in canonical bijection with the valuation rings of \(K\) of Krull dimension one (HOW).

8) Let \(\Gamma, \preceq\) be a totally ordered group. Prove/answer the following:
   a) \(\Gamma, \preceq\) is embeddable in \(\mathbb{R}, +, \leq\) iff \(\Gamma\) has no proper convex subgroups.
   b) A subgroup \(\Gamma \subseteq \mathbb{R}, +\) is discrete iff \(\Gamma\) is cyclic, i.e., \(\Gamma = a\mathbb{Z}\) for some \(a \in \mathbb{R}\).

   What are the discrete subgroups of the multiplicative group \(\mathbb{R}^\times\)?

**Examples: The Gauss valuation.** Let \(K, v\) be a fixed valued field.

• Let \(K[t]\) be the polynomials in the variable \(t\) over \(K\). For \(f = \sum_i a_i t^i \in K[t]\) define
\[ w_t(f) = \max_i v(a_i) \in vK, \text{ and } w_t(f/g) = w_t(f) - w_t(g) \text{ for } f, g \in K[t]. \]

9) Show that \(w_t\) is a valuation on \(K(t)\) satisfying: \(w_t K(t) = vK\) and \(\kappa(w_t) = \kappa(v)(t)\).
The valuation \(w_t\) is called the **Gauss valuation** of \(K(t)\) defined by \(v\) and \(t\).

10) Let \(F = K(T)\), where \(T = (t_1, \ldots, t_d)\). What is the Gauss valuation \(w_T\)?

• Let \(vK \subseteq G\) with \(\Gamma\) totally ordered, \(\gamma \in G/vK\) non-torsion. For \(f = \sum_i a_i t^i \in K[t]\) set:
\[ w_\gamma(f) = \max_i v(a_i) + i\gamma, \text{ and } w_\gamma(f/g) = w_\gamma(f) - w_\gamma(g) \in \Gamma \text{ for } f, g \in K[t]. \]

11) Show that \(w_\gamma\) is a valuation on \(K(t)\) satisfying: \(w_\gamma K(t) = vK + \gamma\mathbb{Z}\) and \(\kappa(w_t) = \kappa(v)\).
The valuation \(w_\gamma\) is called the **Gauss valuation** of \(K(t)\) defined by \(v\) and \(\gamma\).

12) Let \(F = K(T)\), where \(T = (t_1, \ldots, t_d)\), and \(G := (\gamma_1, \ldots, \gamma_d)\) from \(\Gamma\) which are linearly independent in \(\Gamma/vK\). What is the Gauss valuation \(w_G\)?

• Give the generalization of the above, by combining the two situations.

13) Describe all the valuations of \(\mathbb{Q}_p(t)\) which prolong the \(p\)-adic valuation \(v_p\) of \(\mathbb{Q}_p\).

14) Viewing all the fields below as abstract fields (endowed with no extra structure), describe:

   a) \(\text{Aut}(\mathbb{R}), \text{Aut}(\mathbb{C}), \text{Aut}(\mathbb{Q}_p), \text{Aut}(\overline{\mathbb{Q}_p})\).
   b) \(\text{Isom}(\mathbb{R}, \mathbb{Q}_p), \text{Isom}(\mathbb{Q}_p, \mathbb{Q}_q), \text{Isom}(\mathbb{C}, \overline{\mathbb{Q}_p}), \text{Isom}(\overline{\mathbb{Q}_p}, \overline{\mathbb{Q}_q}).\)