Due: Wed, Oct 7, 2020, at 3:00PM

Math 202 / Midterm 1 (two pages)

Academic Integrity Statement:

My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this Math 202 exam. That means, among other things, that you are allowed to: (a) discuss the problems of the midterm with your colleagues, but not work out solutions together; (b) ask any member the Math Dept about hints to the exam problems, but you must first mention to her/him that it goes about problems on a take home exam; (c) consult resources (books, Internet, etc.), but you must design/write down your own proofs.

Name (printed): __________________________ Signature: __________________________ Date ________

Note: There are 8 (eight) problems on this exam.

Points: 15 points for each problem (extra and/or partial credit possible).

Grading: 60 < C−, C, C+ ≤ 75 < B−, B, B+ ≤ 90 < A−, A, A+

Procedures: Write your name (printed) and sign the above Academic Integrity Statement. Indicate clearly the number of each problem you work out. Your submission should/can be a photocopy/xerox/scan of the actual work.

• Recall: A complete proof must contain all the necessary explanations/steps, and in order to disprove an assertion you must give a counterexample showing that the assertion is not true.

1) Consider the assertion in plain English:

“Every natural number less than 1024 is a sum of four squares of natural numbers.”

a) Write the above assertion using quantifiers.

b) What is the negation of the above assertion in plain English.

c) Write the negation of the above assertion using quantifiers.

d) Is the above assertion true?

2) Let $A, B, C, D, E$ be given sets. Using the ZF Axioms, show that there exists a set $X$ whose elements are precisely the following sets: $A \cup B$, $C \setminus D$, $E \times B$, $A \cup E$, $A \cup \emptyset$. Further, answer the following:

a) What are the possibilities for the cardinality of $X$?

b) Is $A$ always an element of $X$?

c) Can $X$ be the empty set?

[Hint. Write down the subsets of $X$ which have one element, and analyze what are these sets for particular $A, B, C, D, E$, e.g., $A = B = \ldots$, etc . . .]
3) Let \( A, B \) be non-empty sets, and for maps \( f : A \to B \), and subsets \( A' \subset A, B' \subset B \), recall the definitions of \( f(A') \subset B, f^{-1}(B') \subset A \). Prove/disprove/disprove/answer the following:

a) \( f : A \to B \) is injective iff for all \( A', A'' \subset A \) one has \( f(A' \cap A'') = f(A') \cap f(A'') \).

b) \( f : A \to B \) is bijective iff for all \( A' \subset A, B' \subset B \) one has: \( |f(A')| = |A'|, |f^{-1}(B')| = |B'| \).

[Hint. If \( f(x_1) = f(x_2) \) and \( A' = \{x_1, x_2\} \), then \( |f(A')| = 1 \) (WHY), etc... If \( y \notin f(A) \) and \( B' = \{y\} \), then \( f^{-1}(B') = \emptyset \) (WHY), etc...]

4) Let \( A, \leq \) and \( A', \leq' \) be totally ordered sets, and \( f : A \to A' \) be a strictly increasing map, i.e., \( \forall x_1, x_2 \in A \) one has: \( x_1 \leq x_2 \Rightarrow f(x_1) \leq' f(x_2) \). Prove/disprove/answer:

a) If \( |A| = 19 \), then \( A' \) has more than 5 elements. What is the converse of this assertion, and is the converse true? (Justify your answer!)

b) Suppose that \( |A| = 3 \) and \( |A'| = n \) is a finite set. How many maps \( f : A \to A' \) are there? (Justify your answer!)

5) Define on \( \mathbb{N} \) the relation \( \sim \) by:

\[ m \sim n \overset{\text{def}}{\iff} m \text{ and } n \text{ give the same remainder under division by 99.} \]

Prove/disprove/disprove the following:

a) \( \sim \) is an equivalence relation on \( \mathbb{N} \).

b) What are the equivalence classes of 0, 1, 101?

6) Prove the following:

a) An arbitrary product of odd numbers is an odd number.

b) Let \( n \) be a natural number such that \( \gcd(n, 9) = 3 \). Then \( \sqrt{n} \) is not a rational number.

[Hint to b): \( \gcd(n, 9) = 3 \Rightarrow 3|n, \text{ but } 9
\notin n, \text{ and continue by contradiction...} \]

7) Let \( m, n \in \mathbb{N} \) be given. Prove/disprove/disprove/answer the following:

a) For all prime numbers \( p \) one has: \( p|((60m^5n^3) \text{ iff } p|(90mn^7)). \)

b) \( m, n \) are odd iff \( 8|(n^4m^4 - 1) \).

c) Is it true that \( m, n \) are odd iff \( 16|(n^4m^4 - 1) \)?

[Hints. To a): Use that \( p|(k \cdot l) \text{ iff } p|k \text{ or } p|l, \text{ etc... To b), c): } k \in \mathbb{N} \text{ is odd iff } k = 2l + 1, \text{ etc...} \]

8) For fixed natural numbers \( a, b, c \in \mathbb{N} \), define on \( \mathbb{N} \) a new addition by \( x \oplus y = x + y - 1 \), and a new multiplication by \( x \odot y = xy + ax + by + c \).

a) Find all \( a, b, c \in \mathbb{N} \) such that \( + \) and \( \odot \) are associative, commutative, have neutral elements 0\( \oplus \), and 1\( \odot \), and \( \odot \) is distributive w.r.t. \( \oplus \).

b) Do the equations \( x \oplus x \oplus 3 \odot x = 1 \odot \) and \( x \odot x \oplus 3 \odot x = 0 \oplus \) have solutions in \( \mathbb{N} \)?