Math 202 (Proofs – Analysis)

Pre-HW: Try to solve . . . (2 pages)

• Solve as much as you can of the problems below. For each problem make explicit the precise hypotheses you are using to tackle the problem.

• Recall that in order to disprove a particular assertion $P$, one must give an example in which the hypothesis of $P$ is satisfied, but the conclusion of $P$ does not hold.

Example:

• Assertion $P$: Every natural number is a sum of three squares of natural numbers.

Solution. The assertion is false, because 7 is not a sum of three squares of natural numbers.

1) Let $m, n$ be positive natural numbers. Prove/disprove:

$mn$ is an odd number if and only if both $m$ and $n$ are odd numbers.

Is the same true for the product $n_1 \ldots n_{100}$ of any hundred positive natural numbers?

Hint: What is the general form of an odd natural number?

2) What is the remainder of the division of $3^{2019}$ by 5, respectively by 16?

Hint: $3^4 = 81$ has remainder 1 when divided by 5, etc.

3) Prove/disprove the following: If a natural number $n$ equals the cube of another natural number $m$, then $m$ and $n$ are divisible by the same prime numbers.

4) Let $n$ be a positive natural number divisible by 3, but not divisible by 9. Prove that $\sqrt{n}$ is not a rational number.

Hint: By contradiction, suppose that $\sqrt{n}$ is rational, hence $\sqrt{n} = k/l$ for some relatively prime natural numbers $k, l$ (WHY). Then $l^2n = k^2$, hence 3 divides $k$ (WHY). Thus $k = 3k'$, hence $l^2n = 9k'^2$ (WHY). Therefore, $l$ is divisible by 3 (WHY), etc.

5) Find all the real numbers $x$ such that $x^6 + x^4 - 2x^2 + 1 = 0$.

Find all the pairs of real number $(x, y)$ such that $x^2 - 5xy^2 + 7y^4 = 0$.

6) Let $c > 0$ be a positive real number. Prove/disprove:

There exists a natural number $n$ such that $nc > 100$.

7) Prove/disprove that the function $\sin(x)$ cannot be written as a polynomial function $f(x)$.

The same question about the exponential function $2^x$.

8) List these numbers from smallest to largest:

$2^{121}$, $9^{55}$, $7^{88}$, $N :=$ (number of seconds since the birth of our universe —Google it!)

9) Using the definitions of the set operations $\cup$, $\cap$, $\setminus$, $\times$, prove the following:

a) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ and $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

b) De Morgan laws: $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$, $C \setminus (A \cap B) = C \setminus A \cup C \setminus B$.

c) $(A \cup B) \times C = (A \times C) \cup (B \times C)$ and $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

Hint: Recall that to prove that two sets are equal $X = Y$ one has to prove that $x \in X \Rightarrow x \in Y$, and $y \in Y \Rightarrow y \in X$.

10) Let $f : A \to B$, $g : B \to C$ be maps, and consider $g \circ f : A \to C$. Prove/disprove:
a) If $f$ and $g$ are injective (reps. surjective), then $g \circ f$ is injective (reps. surjective).

b) If $f$ and $g$ are bijective, then $g \circ f$ is bijective.

11) Prove that there is no smallest strictly positive real number $c > 0$.

[Q**uestions:** What is a real number? What does “smallest” mean?]

12) Knowing that the sum of the angles (in radians) of a triangle is $\pi$, prove that the sum of the angles (in radians) of a convex $n$-gon is $(n - 2)\pi$. What about non-convex $n$-gons?

[Q**uestions:** What is a radian? What is a (non-convex) $n$-gon?]

13) Prove that $S_1(n) := 1 + \cdots + n = n(n + 1)/2$ for all natural numbers $n > 0$. What is the generalization of this in terms of arithmetic progressions? (Google it!)

Are there similar “closed formulas” for $S_2(n) := 1^2 + \cdots + n^2$, and/or $S_k(n) := 1^k + \cdots + n^k$ for every natural number $k$?

14) The Fibonacci sequence (Google it!) is defined recursively (what is that?), as follows:

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \quad \text{for } n > 1.$$ 

Find real numbers $A, B, \alpha, \beta$ such that $F_n = A\alpha^n + B\beta^n$ for all $n > 0$, and prove the resulting “closed formula” by induction. Are the numbers $A, B, \alpha, \beta$ unique?

• Are there similar “closed formulas” for all sequences $(x_n)_n$ defined recursively?

Recall that the binomial coefficients $\binom{n}{m}$ are defined by $\binom{k+l}{k} = \frac{(k+l)!}{k!l!} = \binom{k+l}{l}$ for all natural numbers $k, l$, where $0! = 1$ by definition.

15) Prove the following, e.g. by induction (on what?):

a) $\binom{n}{m}$ equals the number of subsets with $m$ elements of the set $\{1, \ldots, n\}$.

b) The binomial formula holds: $(x + y)^n = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} y^m$

[Q**uestion:** What are the precise hypothesis on $x, y$ and the operations $+$ and $\cdot$ for the above formula to hold?]

H**int** to b): As a first step, prove that $\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$ for all $m, n > 0$, and use this in the induction hypothesis, etc.

16) Show that $2^n = \sum_{m=0}^{n} \binom{n}{m}$. Are there similar formulas for $3^n$, $4^n$, etc.?

17) Let $a_1, \ldots, a_n > 0$ be real numbers. Prove the famous mean inequalities (Google it!):

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} \geq \frac{1}{a_1} + \cdots + \frac{1}{a_n}$$

N**ote:** There are more general inequalities, the so called Jensen inequalities (Google it!).

H**int:** Let $\mu_n(a_1, \ldots, a_n)$ be one of the means above. Prove (by induction?) the following properties of $\mu_n$:

(i) $\mu_{n+1}(a_1, \ldots, a_n, \mu_n(a_1, \ldots, a_n)) = \mu_n(a_1, \ldots, a_n)$. (ii) $\mu_{2k+1}(a_1, \ldots, a_{2k+1}) = \mu_2(\mu_k(a_1, \ldots, a_k), \mu_k(a_{2k+1}, \ldots, a_{2k+1}))$.

Prove the mean inequalities by induction, first on $n = 2^k$, then for all $n < 2^k$, etc.

18) Obviously, there are natural numbers $n$ which cannot be described using less than 1000 words (say, in English). Let $n_0$ be the smallest such number, in other words, one has that:

$n_0$ is the smallest natural number which cannot be described using less than 1000 words.

OTOH, the above is a description of $n_0$ with less than 1000 words!?... What is wrong here?