Due: April 19, 2024

Math 6030 / Problem Set 10 (two pages)

Miscellaneous

Let R be a comm. ring, $f_1, \dots, f_n \in R$ be s.t. $(f_1, \dots, f_n) = R$ and $R_{f_i} = R[\frac{1}{f}] = R_{\Sigma_{f_i}}$ be the ring of fraction of R w.r.t. $\Sigma_{f_i} = \{f^n \,|\, n \in \mathbb{N}\}$. Equivalently, the Zariski open subsets $D_{f_i} := \{\mathfrak{p} \in \mathrm{Spec}(R) \,|\, f_i \notin \mathfrak{p}\} \subset \mathrm{Spec}(R)$ constitute on open covering of $\mathrm{Spec}(R)$ (WHY). Note also setting $f_{ji} := f_j f_i$, one has $D_{f_{ji}} = D_{f_i} \cap D_{f_j}$ for all i, j (WHY). For an R-module M, let $M_{f_i} := M \otimes_R R_{f_i}$ be the corresponding module of fractions.

1) In the context above, prove the following useful/famous:

Globalization Lemma: *Let $m_i \in M_{f_i}$ be such that $\forall\, j, k$ and all $\mathfrak{p} \in D_{f_{jk}}$ one has: $\frac{m_j}{1} = \frac{m_k}{1}$ in $M_\mathfrak{p}$. Then there exists $m \in M$ such that $m_i = \frac{m}{1}$ inside M_{f_i}.*

Hint: Prove the following:

- There exist properly chosen powers $f'_i := f_i^{m_i}$, and elements $m'_i \in M$ such that $m_i = \frac{m'_i}{f'_i}$.
- Show that w.l.o.g., we can suppose replace f_i by f'_i for all i; thus have: $\frac{m'_j}{f_i} = \frac{m'_k}{f_k}$ for all j, k.
- From the equality $\frac{m'_j}{f_i} = \frac{m'_k}{f_k}$, deduce that $\exists\, n$ (sufficiently large) such that $(f_j f_k)^n f_k\, m'_j = (f_j f_k)^n f_j\, m'_k$ for all j, k.
- Replace m'_i by $m'' := f_i^n m'_i$ for all i, and conclude that $m_i = \frac{m'_i}{f'_i} = \frac{m''_i}{f_i^n}$.
- Show that setting $g_i = f_i^{n+1}$ we have: $(D_{g_i})_i$ is an open covering of $\mathrm{Spec}(A)$, and $g_k m''_j = g_j m''_k$ for all j, k.
- Finally, $\exists\, a_i \in A$ such that $\sum_i a_i g_i = 1$. Show that $m := \sum_i a_i m''_i$ does the job.

2) Let R be a commutative ring with 1, and view $R[t]$ as a ring extension of R.
For an ideal $\mathfrak{a}$ of R, let $\mathfrak{a}[t]$ be the set of all the polynomials with coefficients from $\mathfrak{a}$.

a) Show that $\mathfrak{a}[t]$ is an ideal of $R[t]$, and that $\mathfrak{a}[t] \in \mathrm{Spec}(R[t])$ iff $\mathfrak{a} \in \mathrm{Spec}(R)$.

b) Prove/disprove: $\mathcal{N}(R[t]) = \big(\mathcal{N}(R)\big)[t]$, where $\mathcal{N}(\cdot)$ denogtes the nil-radical.

c) Show that $\mathrm{Krull.dim}(R[t]) \geqslant \mathrm{Krull.dim}(R) + 1$.
Prove/disprove: The above inequality is actually an equality.

d) The same question form c) in the case $R = \mathbb{Z}$, or more general, R Noetherian.

Integral ring extensions

Recall the notations: $S|R$ for a ring extension, $\widetilde{R}|R$ is the integral closure of R in S, and $\tilde{\mathfrak{a}} \subset S$ be the set of all $x \in S$ which are integral over $\mathfrak{a} \in \mathcal{I}d(R)$. Let $\mathfrak{b} \in \mathcal{I}d(S)$ be proper ideals and for multiplicative systems $\Sigma \subset R$ consider the resulting $R_\Sigma \to \widetilde{R}_\Sigma \to S_\Sigma$.

3) Prove/disprove/answer:

a) $\tilde{\mathfrak{a}}$ equals the nil-radical $\tilde{\mathfrak{a}} = \mathcal{N}(\mathfrak{a}\widetilde{R})$ of $\mathfrak{a}\widetilde{R}$ in $\widetilde{R}$.

b) Set $\mathfrak{a} := \mathfrak{b} \cap R$, $\tilde{\mathfrak{a}} := \mathfrak{b} \cap \widetilde{R}$. Then $\widetilde{R}/\tilde{\mathfrak{a}}$ is integral over $R/\mathfrak{a}$. Does it hold $\widetilde{R/\mathfrak{a}} = \widetilde{R}/\tilde{\mathfrak{a}}$?

c) (i) $R_\Sigma \to \widetilde{R}_\Sigma \to S_\Sigma$ are ring extensions; (ii) $\widetilde{R}_\Sigma$ is integral over R_Σ; (iii) $\widetilde{R_\Sigma} = \widetilde{R}_\Sigma$.

4) Give an example of an integral ring extension $S|R$ for which *Going down* does not hold, i.e., there are prime ideals $\mathfrak{p}_1 \subset \mathfrak{p}_2$ in $\mathrm{Spec}(R)$ and $\mathfrak{q}_2 \in \mathrm{Spec}(S)$ s.t. $\mathfrak{p}_2 = \mathfrak{q}_2 \cap R$, bur there is **no** prime ideal $\mathfrak{q}_1 \subset \mathfrak{q}_2$ s.t. $\mathfrak{p}_1 = \mathfrak{q}_2 \cap R$.

5 A *quadratic number field* is any field extension $K|\mathbb{Q}$ s.t. $[K : \mathbb{Q}] = 2$. Recall that the *integral closure $\mathcal{O}_K$ of $\mathbb{Z}$ in K is the ring of integers of K.* Show the following:

a) For $K|\mathbb{Q}$ quadratic there a unique square free $\exists\, d \in \mathbb{Z}$, $d \neq 1$ such that $K = \mathbb{Q}[\sqrt{d}\,]$.

b) Compute the ring of integers $\mathcal{O}_K$ of $K = \mathbb{Q}[\sqrt{d}\,]$ for $d = -1, \pm 2, \pm 3$.

c) Do you recognize the general rule which gives $\mathcal{O}_K$?

6) Let $k = \overline{k}$ be algebraically closed field, $R = k[t]$ be the polynomial ring, $S := k[t_1, t_2]/\mathfrak{a}$, where $\mathfrak{a} = (t_2^2 - t_1^3 + t_1^2 + 6)$. Hence setting $x_i := t_i (\mathrm{mod}\, \mathfrak{a})$, $i = 1, 2$, have $S = k[x_1, x_2]$ (WHY).

a) Show that S is an integral domain.

b) Compute the integral closure $\widetilde{S}$ of S in $L = \mathrm{Quot}(S)$.

c) For $\varphi : R \to S$, $t \mapsto x_1$, prove/disprove: S becomes an integral ring extension via φ.

d) Compute the fibers of $\varphi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(S)$ and of $\imath^* : \mathrm{Spec}(\widetilde{S}) \to \mathrm{Spec}(S)$, where $\imath : S \to \widetilde{S}$ is the inclusion.

7) The same questions as above in the case $\mathfrak{a} = (t_2^2 - t_1^3)$.

• Recall the context of the **Key Lemma** concerning the *special change of variables* as follows: k is an arbitrary field, $\underline{\boldsymbol{X}} = (X_1, \dots, X)$ are k-independent variables, $\underline{\boldsymbol{\imath}} = (i_1, \dots, i_n) \in \mathbb{N}^n$ are multi-indices, and $\|\underline{\boldsymbol{\imath}}\| = i_1 + \cdots + i_n$ is the norm of $\underline{\boldsymbol{\imath}}$. Further, $p(\underline{\boldsymbol{X}}) = \sum_{\underline{\boldsymbol{\imath}}} a_{\underline{\boldsymbol{\imath}}} \underline{\boldsymbol{X}}^{\underline{\boldsymbol{\imath}}} \in k[\underline{\boldsymbol{X}}]$ is a non-zero polynomial of total degree $d \geqslant 0$. Hence $p(\underline{\boldsymbol{X}}) = \sum_{m=0}^{d} p_{(m)}(\underline{\boldsymbol{X}})$, where $p_{(m)}(\underline{\boldsymbol{X}}) = \sum_{\|\underline{\boldsymbol{\imath}}\|=m} a_{\underline{\boldsymbol{\imath}}} \underline{\boldsymbol{X}}^{\underline{\boldsymbol{\imath}}}$ is the homogeneous part of degree m of $p(\underline{\boldsymbol{X}})$.
Finally for $d >$, consider the change of variables:

(I) $X_n = a_n X'_n$, $X_i = X'_i + a_i X'_n$ with $a_i \in k$, $1 \leqslant i \leqslant n$.

(II) $X_n = X_n'^{m_n}$, $X_i = X'_i + X_n'^{m_i}$ with $m_i \in \mathbb{N}$, $1 \leqslant i \leqslant n$.

8) For $p(\underline{\boldsymbol{X}}) = \sum_{\underline{\boldsymbol{\imath}}} a_{\underline{\boldsymbol{\imath}}} \underline{\boldsymbol{X}}^{\underline{\boldsymbol{\imath}}}$ non-constant of degree d, set $q(\underline{\boldsymbol{X}}') := p(\underline{\boldsymbol{X}})$. Try to prove:

a) If $X \subset k$ is an infinite subset, there are $a_1, \dots, a_n \in X$, $a \in k^\times$ such that

$$q(\underline{\boldsymbol{X}}') = a X_n'^d + \big(\text{terms in which } X'_n \text{ has exponent } < d\big)$$

b) If $X \subset \mathbb{N}$ is an infinite subset, there are $m_1 \ll \cdots \ll m_n$ in X, $a_{\underline{\boldsymbol{\imath}}} \neq 0$ such that

$$q(\underline{\boldsymbol{X}}') = a_{\underline{\boldsymbol{\imath}}} X_n'^{n_0} + \big(\text{terms in which } X'_n \text{ has exponent } < n_0\big)$$

[**Hint**: To a): First, $q(\underline{\boldsymbol{X}}')$ has total degree d (WHY), and second, the coefficient of X'^d is $p(a_1, ..., a_n)$ (WHY), etc.

To b): Let $\prec$ be the lexicographic ordering of $\mathbb{N}^d$ **(what is that?!)**. Let $I \subset \mathbb{N}^n$ be the set of all $\underline{\boldsymbol{\imath}}$ s.t. $a_{\underline{\boldsymbol{\imath}}} \neq 0$, and $\underline{\boldsymbol{\imath}}_0 = \max(I)$. Next show (by induction of n) that for properly chosen $m_1 \ll \cdots \ll m_n$ in X, the map $\varphi : [0, \dots, d]^n \to \mathbb{N}$, $\underline{\boldsymbol{\imath}} \mapsto \sum i_j m_j$ preserves the total ordering. Hence $n_0 := \varphi(\underline{\boldsymbol{\imath}}_0) = \max(\varphi(I))$ is the maximal degree of X'_n in $q(\underline{\boldsymbol{X}}')$, and it is attended only at $\underline{\boldsymbol{\imath}} = \underline{\boldsymbol{\imath}}_0$ (WHY), etc. . . .]