

Math 6030 / Problem Set 12 (two pages)

Prolongations of valuations. Let $R = R_v$ be a valuation ring of a field K with canonical valuation v , valuation ideal \mathfrak{m}_v , valuation group $vK = K^\times/R_v^\times$ and residue field $\kappa_v = R_v/\mathfrak{m}_v$. Let $L|K$ be a field extension and R_w be a valuation ring of L with canonical valuation w , valuation ideal \mathfrak{m}_w , value group $wL = L^\times/R_w^\times$ and residue field κ_w .

Recall: We say that: w prolongs v if $vK \subset wL$ and $v = w|_K$, denoted $w|v$, and that R_w prolongs R_v if R_w, \mathfrak{m}_w dominates R_v, \mathfrak{m}_v , i.e., $\mathfrak{m}_v = \mathfrak{m}_w \cap R$, denoted $R_w|R_v$. If so, define: $e(w|v) := (wL : vK)$ the ramification index, and $f(w|v) := [\kappa_w : \kappa_v]$ the residue degree of $w|v$.

1) In the above notation, prove/disprove/answer:

- a) One has: $w|v$ iff $R_w|R_v$ iff $R^\times = K \cap R_w^\times$.
- b) Suppose that $w|v$ or equivalently, $R_w|R_v$.
 - If $u_i \in L$ and $w(u_i) \neq w(u_{i'})$ for all $i \neq i'$, then $w(\sum_i u_i) = \min_i w(u_i)$.
 - If $x_j \in R$ and $(\bar{x}_j)_j$ are κ_v lin. indep. in κ_w , then $(x_j)_j$ are K -lin. indep. in L .

2) Suppose that $R_w|R_v, R_{w_l}|R_v, l \leq n$ be distinct. In the above notation/context, TFH:

I) **The (weak) Fundamental Inequality.**

Let $(y_i)_i$ in L and $(x_j)_j$ in R_w be s.t. $(w(y_i))_i$ are distinct in $wL \rightarrow wL/vK$ and $(\bar{x}_j)_j$ are κ_v -linearly independent in κ_w . Then $(x_i y_j)_{i,j}$ is K -linearly independent in L .

II) **The Fundamental Inequality.** One has $\sum_l e(w_l|v)f(w_l|v) \leq [L : K]$.

3) In the above context, let $L|K$ be algebraic, $S|R_v$ be the integral closure of R_v in L , and $X_v = \{w \in \text{Val}(L) \mid w|v\}$ be the set of prolongations of v to L . Prove/disprove/answer:

- a) The map $X_v \rightarrow \text{Max}(S), \mathfrak{m}_w \mapsto \mathfrak{n} := \mathfrak{m}_w \cap S$ is a well defined bijection and $R_w = S_{\mathfrak{n}}$.
- b) For every $w \in X_v$ one has: wL/vK is a torsion group, and $\kappa_w|\kappa_v$ is algebraic.

Moreover, if $L = \bar{K}$, then wL is divisible and $\kappa_w = \bar{\kappa}_w$. Does the converse hold?

[Hint to a): For $x \in L^\times$, let $\text{Mipo}_K(x) = t^n + \dots + a_0 \in K[t]$. If i_0 is maximal s.t. $v(a_{i_0}) = \min_i v(a_i)$, set $b_i = a_{i+i_0}/a_{i_0}$ for $i \leq n - i_0$, $c_j = a_j/a_{i_0}$ for $j < m$. Then $p(t) = \sum_i b_i t^i \in 1 + t\mathfrak{m}_v[t]$ (WHY) and $q(t) = \sum_j c_j t^j \in R_v[t]$ (WHY). Further, $p(x) = \frac{1}{x} q(\frac{1}{x})$ (WHY), and $p(x) \in 1 + \mathfrak{m}_v[x]$ (WHY). Next, $\forall R_w|R_v$ have: $x \in R_w \Rightarrow p(x) \in R_w$, thus $q(\frac{1}{x}) \in R_w$; $\frac{1}{x} \in R_w \Rightarrow q(\frac{1}{x}) \in R_w$, thus $p(x) \in R_w$ (WHY). Conclude: $\forall x \in K$ have $p(x), q(\frac{1}{x}) \in S$ (WHY). Now suppose that $x \in \mathfrak{m}_w$. Then $p(x) \in 1 + \mathfrak{n}$ (WHY) and $x p(x) = q(\frac{1}{x})$ implies: $q(\frac{1}{x}) \in \mathfrak{m}_w \cap R = \mathfrak{n}$ (WHY). Thus $x = p(x)^{-1} q(\frac{1}{x}) \in \mathfrak{n}$ (WHY), etc.

4) In the above context/notation, let R_v be a DVR and $[L : K]$ be finite. Prove/disprove:

- a) S^+ is a finite R_v -module iff the fundamental equality $\sum_l e(w_l|v)f(w_l|v) = [L : K]$ holds.
- b) If $L|K$ is finite separable, the fundamental equality $\sum_l e(w_l|v)f(w_l|v) = [L : K]$ holds.

More about the integral closure. Let $\widehat{K} := k((t))$ endowed with \hat{v} . Then $\text{td}(\widehat{K}|k) = \infty$ (WHY), and for $t, u \in \widehat{K}$ alg. indep. over k , set $K = k(t, u^{p^e})$, $L = k(t, u)$, and $w = \hat{v}|_L, v = \hat{v}|_K$.

5) In the above notation, let $\text{char}(k) = p > 0$. Prove/disprove:

- a) $L|K$ is a purely inseparable of degree p^e .
- b) w is the unique extension of v to L and $e(w|v) = 1 = f(w|v)$.
- c) $R_w|R_v$ is the integral closure of R_v in L , but R_w is not a finite R_v -module.

6) Complete the proof of the assertion from the class:

Let $R = k[x_1, \dots, x_n]$ is a finitely generated domain over a field k , $K = \text{Quot}(R)$, $L|K$ is a finite field extension, and $S|R$ be the integral closure of R in L . Then S is a finite R -module.

[Hint: Use the explanations from the class to reduce to the case $R_0 = k[t_1, \dots, t_d]$, hence $K_0 = k(t_1, \dots, t_d)$ and $L_0|K_0$ purely inseparable. Conclude by induction on $[L_0:K_0]$ as follows: If $\alpha^p = p(t_1, \dots, t_d)$, then $\alpha \in K_1 = k_1(u_1, \dots, u_d)$, where $k_1|k$ is the finite field extension obtained by adjoining the p -th roots of all the coefficients of $p(t_1, \dots, t_d)$ and $u_i^p = t_i$ for $1 \leq i \leq d$, etc. ...]

7) Describe the decomposition groups of the prime ideals $\mathfrak{p} \in \text{Spec}(R)$ in $S|R$ below:

- a) $R = \mathbb{Z}$, S the ring of algebraic integers in $K = \mathbb{Q}[\sqrt{d}]$ with $1 < |d| < 7$.
- b) $R = \mathbb{Z}$, S the ring of algebraic integers in $K = \mathbb{Q}[\zeta_3, \sqrt[3]{2}]$, $\mathfrak{p} = 2\mathbb{Z}, 3\mathbb{Z}, 5\mathbb{Z}, 7\mathbb{Z}$.

More about fractional ideals.

8) Let R be a Noetherian domain, $\text{Min}(R)$ minimal prime ideals $\mathfrak{p} \neq (0)$. Prove/disprove/answer:

- a) An ideal $\mathfrak{a} \subset R$ is invertible iff $\text{Spec}(\mathfrak{a}) \subset \text{Min}(R)$, and all $\mathfrak{p} \in \text{Min}(\mathfrak{a})$ are invertible.
- b) If $\mathfrak{p} \in \text{Spec}(R)$ is invertible, then $\mathfrak{p} \in \text{Min}(R)$ and $R_{\mathfrak{p}}$ is integrally closed. Conversely?
- c) R is integrally closed iff all $\mathfrak{p} \in \text{Min}(R)$ are invertible.

Conclude: $\text{Div}(R) = \bigoplus_{\mathfrak{p}} \mathbb{Z} \mathfrak{p}$ with $\mathfrak{p} \in \text{Min}(R)$ invertible.

[Hint to a), b): Use Krull Principal Ideal Thm combined with the Lemma from the proof of the Characterization Thm, etc. ...]

9) Prove the following basic facts about Dedekind domains R :

- **Gauss Lemma for Dedekind domains R :** For $f = a_n t^n + \dots + a_0 \in R[t]$ let $c(f) := (a_0, \dots, a_n) \in \mathcal{I}d(R)$ be the content of f . Then one has $c(fg) = c(f)c(g)$.
- For $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{I}d(R)$ one has: $\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} \cap \mathfrak{b}) + (\mathfrak{a} \cap \mathfrak{c})$, $\mathfrak{a} + (\mathfrak{b} \cap \mathfrak{c}) = (\mathfrak{a} + \mathfrak{b}) \cap (\mathfrak{a} + \mathfrak{c})$.
- A finite R -module M is flat iff M is torsion free.
- Let M be a finite torsion R -module. There are $(\mathfrak{p}_i)_i, (e_i)_i, i \in I$ finite s.t. $M = \bigoplus_i R/\mathfrak{p}_i^{e_i}$.
- For $N \subset R^N$ an R -submodule, $\exists \mathfrak{a} \in \mathcal{I}d(R), N_0 \subset N$ R -free such that $N = N_0 \oplus \mathfrak{a}$.

[Hint: Localize at each $\mathfrak{p} \in \text{Max}(R)$, and use the fact that two modules are equal, morphisms are injective/surjective, etc. iff the corresponding assertions hold everywhere locally, etc.... For the last assertion, show that $\mathfrak{a} \oplus \mathfrak{b} \cong_R R \oplus \mathfrak{c}$, etc....]