Due: Feb 16, 2024

Math 6030 / Problem Set 3 (two pages)

More about the functor $\mathrm{Spec}(R)$**.**

Let T be a topological space. Recall: A subset $X \subset T$ is irreducible, if X is closed and cannot be represented as $X = X_1 \cup X_2$ with $X_1, X_2 \subset T$ distinct closed subsets. Further, an irreducible component of T is any maximal irreducible subset in T. A subset $X \subset T$ is disconnected if $X \subset D_1 \cup D_2$ with $D_1, D_2 \subset X$ open and disjoint and $X \cap D_1, X \cap D_2 \neq \emptyset$. Further, a connected component of T is a maximal subset of T which is connected. A subset which is not disconnected is called connected. Finally, recall that a (commutative) ring R is called regular von Neumann if $\forall\, r \in R\ \exists\, r' \in R$ s.t. $e_r := r'r$ is an idempotent s.t. $re_r = r$.

1) For $\mathfrak{p} \in \mathrm{Spec}(R)$, let $X_{\mathfrak{p}} = \overline{\{\mathfrak{p}\}} \subset \mathrm{Spec}(R)$ be the topological closure. Prove/disprove:

a) $\mathfrak{q} \in X_{\mathfrak{p}}$ iff $\mathfrak{p} \subset \mathfrak{q}$ and $X_{\mathfrak{p}} \subset \mathrm{Spec}(R)$ is irreducible.
In particular, the only closed points of $\mathrm{Spec}(R)$ are $\mathfrak{m} \in \mathrm{Max}(R)$.

b) $\mathrm{Spec}(R)$ is irreducible iff $\mathrm{Min}(R) = \{\mathfrak{p}_0\}$ consists of one point.

c) The irreducible components of $\mathrm{Spec}(R)$ are precisely $X_{\mathfrak{p}_0} \subset \mathrm{Spec}(R)$ with $\mathfrak{p}_0$ minimal.

2) Prove/disprove:

a) $\mathrm{Spec}(R)$ is compact iff $\dim(R) = 0$ iff $\mathrm{Spec}(R)$ is a profinite topological space.

b) Suppose that R is regular von Neumann. Then $\mathcal{N}(R) = (0)$ and $\mathrm{Spec}(R)$ is compact.
Make an educated guess: *Does the converse of the assertion at* b) *above hold?*

c) $\mathrm{Spec}(R)$ is disconnected iff $R = R_1 \times R_2$ iff $\exists\, e \in R$ idempotent $e \neq 1_R, 0_R$.

3) For a morphism $f : R \to S$ in $\mathbf{Rings}^{\mathrm{com}}$ with $f(1_R) = 1_S$, prove/disprove/answer:

a) $f^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a closed immersion iff $f : R \to S$ is surjective.

b) TFAE: (i) $\mathfrak{a}^{ec} = \mathfrak{a}\ \forall\, \mathfrak{a} \in \mathfrak{Id}(R)$; (ii) $f^*\big(\mathrm{Spec}(S)\big) = \mathrm{Spec}(R)$; (iii) $f^*\big(\mathrm{Max}(S)\big) = \mathrm{Max}(R)$.

c) $\mathbf{Rings}^{\mathrm{com}} \rightsquigarrow \mathbf{Top}$, $R \mapsto \mathrm{Spec}(R)$ maps finite products to coproducts.

(∗) Make educated guesses: *Does the same hold correspondingly for*
(i) finite coproducts. (ii) arbitrary products. (iii) projective/inductive limits.

Rings/modules of fractions & Localization.

Recall that all the rings considered here a commutative with identity, and for multiplicative systems $\Sigma \subset R$ we suppose that $0 \notin \Sigma$ and $R_\Sigma = \Sigma^{-1}R$ denotes the Σ ring of fractions of R with structure morphism $\varphi_\Sigma : R \to R_\Sigma$, $a \mapsto \frac{a}{1}$. Make sure that you know/checked the details that the addition and multiplication in R_Σ are well defined and that R_Σ, φ_Σ has the universal property given in class. Further, recall the following:

- $\Sigma_{\mathfrak{p}} := R \backslash \mathfrak{p}$ is multiplicative system $\forall\, \mathfrak{p} \in \mathrm{Spec}(R)$ (WHY).
Further, $\varphi_{\mathfrak{p}} : R \to R_{\mathfrak{p}} := R_{\Sigma_{\mathfrak{p}}}$ is the localization of R at $\mathfrak{p}$.
- $\Sigma_0 := \{r \in R \,|\, r \text{ is not zero divisor}\}$ is a multiplicative system (WHY).
Further, $\varphi_0 : R \to \mathrm{Quot}(R) := R_{\Sigma_0}$ is the total ring of fraction of R.

Given a multiplicative system $\Sigma \subset R$, define its saturation $\tilde{\Sigma} := \{r \in R \,|\, \exists\, s \in \Sigma \text{ s.t. } r \mid s\}$.
Given $\Sigma \subset R$, define the Σ-saturation of $\mathfrak{a}$ by $\tilde{\mathfrak{a}}_\Sigma := \{a \in R \,|\, \exists\, r \in \Sigma \text{ s.t. } ar \in \mathfrak{a}\}$, and denote:

$$\mathfrak{Id}(R)_\Sigma := \{\mathfrak{a} \in \mathfrak{Id}(R) \,|\, \mathfrak{a} \cap \Sigma = \emptyset\}, \quad \operatorname{Spec}(R)_\Sigma = \operatorname{Spec}(R) \cap \mathfrak{Id}(R)_\Sigma.$$

4) Let $\Sigma_1 \subset \Sigma_2 \subset R$ be multiplicative systems. Prove/disprove/answer:

a) There is a canonical ring homomorphism $\varphi_{\Sigma_1\Sigma_2} : R_{\Sigma_1} \to R_{\Sigma_2}$.

b) If $\Sigma_2 = \tilde{\Sigma}$ is the saturation of $\Sigma = \Sigma_1$, then $\varphi_{\Sigma\tilde{\Sigma}}$ is an isomorphism.

c) Make an educated guess: When is $\varphi_{\Sigma_1\Sigma_2}$ an isomorphism?

5) In the above notation, prove/disprove/answer:

a) $\tilde{\mathfrak{a}}_\Sigma \in \mathfrak{Id}(R)$, $\mathfrak{a} \subset \mathfrak{a}_\Sigma$, and $\tilde{\mathfrak{a}}_\Sigma = \tilde{\mathfrak{a}}_{\tilde{\Sigma}}$. Further, $\tilde{\mathfrak{p}}_\Sigma = \mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec}(R)$.

b) For $\mathfrak{a} \in \mathfrak{Id}(R)$ one has $\mathfrak{a} \cap \Sigma = \emptyset$ iff $\tilde{\mathfrak{a}}_\Sigma \cap \Sigma = \emptyset$.

6) For $\varphi_\Sigma : R \to R_\Sigma$, recalling the extension/contraction of ideals, prove/disprove/answer:

a) First, $\mathfrak{a}^e = R_\Sigma$ iff $\mathfrak{a} \cap \Sigma \neq \emptyset$, and second, every $\mathfrak{b} \in \mathfrak{Id}(R_\Sigma)$ is of the form $\mathfrak{b} = \mathfrak{a}^e$.

b) First, $\mathfrak{a}^{ec} = \mathfrak{a}$ iff $\mathfrak{a}$ is Σ-saturated, and second, $\mathfrak{a}^e = \mathfrak{a}'^e$ iff $\tilde{\mathfrak{a}}_\Sigma = \tilde{\mathfrak{a}}'_\Sigma$.

c) $\varphi^*_\Sigma : \operatorname{Spec}(R_\Sigma) \to \operatorname{Spec}(R)_\Sigma \subset \operatorname{Spec}(R)$ is a well defined homeomorphism.

7) Recall $\varphi_0 : R \to \operatorname{Quot}(R) = R_{\Sigma_0}$, and let $x = \frac{a}{r} \in R^\times_{\Sigma_0}$ be given. Prove/disprove:

a) $\tilde{\Sigma}_0 = \Sigma_0$ and $\varphi_\Sigma : R \to R_\Sigma$ is injective iff $\Sigma \subset \Sigma_0$.

b) TFAE: (i) $x \notin R^\times_{\Sigma_0}$; $x \in R_{\Sigma_0}$ is zero divisor in R_{Σ_0}; (iii) $a \in R$ is zero divisor in R.

c) Therefore one has: $\operatorname{Spec}(R_{\Sigma_0}) = \operatorname{Max}(R_{\Sigma_0}) = \operatorname{Min}(R)^e_{\Sigma_0}$, implying that:

$\mathfrak{p} \in \operatorname{Min}(R)$ *iff* $\mathfrak{p}$ *consists of zero divisors only.*

• For a multiplicative system $\Sigma \subset R$ and an R-module M, one has the canonical morphism of $\phi_\Sigma : M \to M_\Sigma = \Sigma^{-1}M$, $x \mapsto \frac{x}{1}$, and M_Σ is canonically an R_Σ-module via $\frac{a}{r} \cdot \frac{x}{s} := \frac{ax}{rs}$. Further, recall the localizations $M_\mathfrak{p}$, $\mathfrak{p} \in \operatorname{Spec}(R)$ and $M_\mathfrak{m}$, $\mathfrak{m} \in \operatorname{Max}(R)$. Finally, consider sequences $(\mathcal{E})$: $M \to N \to P$ of morphisms in R-**Mod**.

8) Prove that $\mathcal{F}_\Sigma : R\text{-}\mathbf{Mod} \rightsquigarrow R_\Sigma\text{-}\mathbf{Mod}$, $M \mapsto M_\Sigma$ is a covariant functor. Prove/disprove:

a) $\mathcal{F}_\Sigma$ is compatible with products and coproducts, inductive/projective limits.

b) If $M \to N \to P$ is exact in R-**Mod**, so is $M_\Sigma \to N_\Sigma \to P_\Sigma$. Hence $\mathcal{F}$ is exact.

c) In the above notation, for a sequence $M \to N \to P$ in R-**Mod**, TFAE:

(i) $M \to N \to P$ is exact.

(ii) $M_\mathfrak{p} \to N_\mathfrak{p} \to P_\mathfrak{p}$ is exact $\forall\, \mathfrak{p} \in \operatorname{Spec}(R)$.

(iii) $M_\mathfrak{m} \to N_\mathfrak{m} \to P_\mathfrak{m}$ is exact $\forall\, \mathfrak{m} \in \operatorname{Max}(R)$.

d) Same question above for arbitrary exact sequences $\cdots \to M_{i-1} \to M_i \to M_{i+1} \to \dots$

Recall: For a free S-module N, its rank $\operatorname{rk}_S(N)$ is the cardinality of an S-basis of N. An R-module M is locally free, if $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module $\forall\, \mathfrak{p} \in \operatorname{Spec}(R)$. Further, if $r \notin \mathcal{N}(R)$, then $\Sigma_r = \{r^n \,|\, n \in \mathbb{N}\}$ is a multiplicative system, and set $R_r := R_{\Sigma_r}$ and $M_r := M_{\Sigma_r}$. Finally, recall that $D_r := \{\mathfrak{p} \,|\, r \notin \mathfrak{p}\}$, $r \in R$ is a basis of Zariski open neighborhoods of $\mathfrak{p}$ (WHY).

9) Prove that an R-module M is locally free iff $M_\mathfrak{m}$ is $R_\mathfrak{m}$-free for all $\mathfrak{m} \in \operatorname{Max}(R)$. Further, for M a finitely generated R-module, prove/disprove/answer:

a) M is locally free iff $\exists\, r_1, \dots, r_n \in R$ s.t. $\cup_i D_{r_i} = \operatorname{Spec}(R)$ and M_{r_i} is R_{r_i}-free $\forall\, i \leqslant n$.

b) Is it true that $\operatorname{rk}_{R_\mathfrak{p}}(M_\mathfrak{p})$ is independent of $\mathfrak{p} \in \operatorname{Spec}(R)$?