Due: March 8, 2024

Math 6030 / Problem Set 5 (two pages)

More about the determinant.

Let M be R-free with $\rbrack \operatorname{rk}(M) = m$. Recall the basics about $R^{m\times m}$ and $\operatorname{End}_R(M)$:

- Every R-basis $\mathcal{A} = (\alpha_1, \dots, \alpha_m)$ of M gives rise to an isomorphism of R-algebras $\Psi_{\mathcal{A}} : \operatorname{End}_R(M) \to R^{m\times m}$; $\varphi \mapsto A_\varphi$, where A_φ is uniquely defined by $\varphi(\mathcal{A}) = \mathcal{A}\cdot A_\varphi$.
- $\mathcal{L}^m_{\text{alt}}(M)$ is a R-free module of rank one, and $\operatorname{End}_R(M), \circ$ acts on $\mathcal{L}^m_{\text{alt}}(M)$ by $\varphi \cdot f := f \circ \varphi^m$, and there is a unique $a_\varphi \in R$ such that $\varphi \cdot f = a_\varphi f$ for all $f \in \mathcal{L}^m_{\text{alt}}$. **Notice** that $f = \mathrm{id}_M \cdot f := a_{\mathrm{id}_R} f$ for all f, thus $a_{\mathrm{id}_M} = 1_R$ (WHY).
- The determinant map $\det : \operatorname{End}_R(M) \to R$, $\varphi \mapsto \det(\varphi) := a_\varphi$ is a morphism of monoids (WHY). And if $A = \Psi_{\mathcal{A}}(\varphi)$, it follows that $\det(A) := \det(\varphi)$ is independent on the R-basis $\mathcal{A}$ defining $\Psi_{\mathcal{A}}$ (WHY).
- **Notice**: If $A \in R^{m\times m}$ has columns $\mathcal{R}_j \in R^{m\times 1}$ and rows $\mathcal{R}_i \in R^{1\times n} = R^n$, then $\det(A_\varphi)$ is the only map $R^{m\times m} \to R$ satisfying/being:

$(*)$ (i) $\det(\boldsymbol{I}_m) = 1_R$; (ii) $\det(A)$ *alternating multilinear in the rows / columns of* A (WHY).

Let $A_{kl} \in R^{(m-1)\times(m-1)}$ be obtained by deleting the k^{th} row and l^{th} column of A, $1 \leqslant k,l \leqslant m$. Then $\Delta_{kl} = \det(A_{kl})$ is the (k,l)-minor of A, and $(-1)^{k+l}\Delta_{kl}$ is the (k,l) cofactor of A, and recall that the matrix $A^* := \big((-1)^{i+j}\Delta_{ji}\big)_{i,j} \in R^{m\times m}$ is the classical adjoint, or adjugate of A.

1) In the above notation, using the properties $(*)$ above of $\det(A)$, prove:

a) *The row / column expansion formula.* Given (k,l) fixed, and $k \neq l$ in case (ii), one has:
(i) $\sum_i (-1)^{i+l} a_{il}\Delta_{il} = \det(A) = \sum_j (-1)^{k+j} a_{kj}\Delta_{kj}$. (ii) $\sum_i (-1)^{i+l} a_{il}\Delta_{ik} = 0 = \sum_j (-1)^{k+j} a_{lj}\Delta_{kj}$.

b) *The inversion formula*: $AA^* = \det(A)\boldsymbol{I}_m = A^*A$.
Hence $A \in \operatorname{GL}_m(R)$ iff $\det(A) \in R^\times$, and if so, $A^{-1} = \det(A)^{-1}A^*$.

Characteristic polynomial. For a commutative ring R with 1_R, set $\tilde{R} := R[t]$, hence $R^{m\times m} \hookrightarrow \tilde{R}^{m\times m}$ canonically. Further, for $A \in R^{m\times m}$ we set $\tilde{A} := t\boldsymbol{I}_m - A$ and say that $p_A(t) := \det(\tilde{A})$ is the characteristic polynomial of A. Further, if $\mathcal{A}$ a basis of a free R-module M and $A_\varphi \in R^{m\times m}$ is the matrix of $\varphi \in \operatorname{End}_R(M)$ in the basis $\mathcal{A}$, then $p_\varphi(t) := p_{A_\varphi}(t)$ is the characteristic polynomial of φ.

2) In the above notation, prove the following:

a) $p_\varphi(t)$ is independent of the concrete basis $\mathcal{A}$ used to define it.

b) The famous **Cayley–Hamilton Theorem**: $p_A(A) = 0_{R^{m\times m}}$ and $p_\varphi(\varphi) = 0_{\operatorname{End}_R(M)}$.

[**Hint** to b): M is a left $\operatorname{End}_R(M)$-module via $f \cdot x = f(x)$ for $f \in \operatorname{End}_R(M)$ (WHY), hence M becomes a left $\tilde{R}$-module via the outer multiplication $f(t) \cdot x := \phi_f \cdot x$, where $\phi_f := f(\varphi) \in \operatorname{End}_R(M)$ (WHY), e.g. one has: $1_{R[t]} \cdot x = x$, $t \cdot x = \varphi(x)$, etc., for all $x \in M$ (WHY). Hence if $\mathcal{A} = (\alpha_1, \dots, \alpha_m)$ is an R-basis of M, $t\cdot\mathcal{A} = \varphi(\mathcal{A}) = \mathcal{A}\cdot A_\varphi$ (WHY), i.e., $\mathcal{A}\cdot\tilde{A}_\varphi = t\cdot\mathcal{A} - \mathcal{A}\cdot A_\varphi = \mathbf{0}_M$, where $\mathbf{0}_M = (0_M, \dots, 0_m)$. Hence have: $\mathbf{0}_M = \mathbf{0}_M\tilde{A}^*_\varphi = (\mathcal{A}\cdot\tilde{A}_\varphi)\tilde{A}^*_\varphi = \mathcal{A}\,(\tilde{A}_\varphi\tilde{A}^*_\varphi) = \mathcal{A}\det(\tilde{A}_\varphi)\boldsymbol{I}_m = p_\varphi(t)\mathcal{A}$. Conclude that setting $\varphi_0 := p_\varphi(\varphi)$, one has: $\mathbf{0}_M = \varphi_0(\mathcal{A})$ (WHY), hence $\varphi_0(\alpha_i) = 0_M$ for all α_i, thus $\varphi_0 = 0_{\operatorname{End}_R(M)}$ (WHY), etc.]

More about $\otimes_R$ and Exactness.

3) Prove/disprove the following:

a) $\otimes_R$ in R-**Mod** is compatible with finite/arbitrary products, respectively coproducts.

b) $\otimes_R$ in R-**Mod** is compatible with inductive, respectively projective limits.

4) Prove/disprove the following:

a) An arbitrary coproduct $\oplus_i M_i$ of R-modules is flat iff M_i is flat for each i. Does the same hold correspondingly for finite/arbitrary products?

b) If $M_1, \dots, M_n$ are flat, so is $M_1 \otimes_R \cdots \otimes_R M_n$. Does the converse hold?

c) If $(M_i, f_{jk})_{i,j\leqslant k}$ is an inductive system of flat R-modules, so is $M = \varinjlim_i M_i$. Does the same hold correspondingly for projective limits?

5) Let S be a commutative R-algebra. Prove/disprove:

a) Being flat is invariant under base change R-**Mod** $\rightsquigarrow$ S-**Mod**, $M \mapsto M_S := B^+ \otimes_R M$, i.e., if M is a flat R-module, so is M_S as S-module.

b) If S^+ is a flat R-module, being flat is invariant under the "restriction of scalars" S-**Mod** $\rightsquigarrow$ R-**Mod**, i.e., if M is a flat S-module, so is M when viewed as R-module.

6) Flatness and Localization. For M in R-**Mod**, prove/disprove/answer the following:

a) If $\Sigma \subset R$ is a multiplicative system, $R_\Sigma \otimes_R M \cong_R M_\Sigma$ canonically (HOW).

b) TFAE: (i) M is flat; (ii) $M_\mathfrak{p}$ is flat $\forall\, \mathfrak{p} \in \mathrm{Spec}(R)$; (iii) $M_\mathfrak{m}$ is flat $\forall\, \mathfrak{m} \in \mathrm{Max}(R)$.

Faithful Flatness. A flat R-module M is faithfully flat, for short (f.f.), if for every sequence of R-modules $(\mathcal{E})\colon\ 0 \to M' \to M \to M'' \to 0$ one has: *$(\mathcal{E})$ is exact iff $M \otimes_R (\mathcal{E})$ is exact.* A commutative flat R-algebra S is called faithfully flat (f.f.), if S^+ is a faithfully flat R-module.

In Problems 7, 8 below, N, P denote arbitrary R-modules, and $\mathfrak{a} \in \mathfrak{Id}(R)$ arbitrary ideals.

7) (Characterization of f.f. R-modules). For a flat R-module M, TFAE:

(i) M is faithfully flat. (ii) $N \otimes_R M = (0)$ iff $N = (0)$. (iii) $\mathfrak{a}M \neq M$ if $\mathfrak{a} \neq (0), R$.
(iii)$'$ $\forall\, \mathfrak{m} \in \mathrm{Max}(R)$, $\mathfrak{m}M \neq M$ if $\mathfrak{m} \neq (0)$. (iv) $f \in \mathrm{Hom}_R(N, P)$ is injective iff $f \otimes \mathrm{id}_M$ is so.

8) (Characterization of f.f. R-algebras).
For a commutative flat R-algebra S, TFAE (compare with HW 3, Problem 3):
(i) S is f.f. (ii) $\mathfrak{a}^{ec} = \mathfrak{a}\ \forall\, \mathfrak{a} \in \mathfrak{Id}(R)$. (iii) $f^*\big(\mathrm{Spec}(S)\big) = \mathrm{Spec}(R)$. (iv) $\mathfrak{m}^e \neq (1_S)\ \forall\, \mathfrak{m} \in \mathrm{Max}(R)$.

More about Hom_R and Exactness

Recall that for an exact sequence $(\mathcal{E})\colon\ 0 \to M' \xrightarrow{\imath} M \xrightarrow{p} M'' \to 0$ TFAE (WHY):
(i) $\exists\, \jmath \in \mathrm{Hom}_R(M, M')$ such that $\jmath \circ \imath = \mathrm{id}_{M'}$. (ii) $\exists\, s \in \mathrm{Hom}_R(M'', M)$ such that $p \circ s = \mathrm{id}_{M''}$.
If so, $\jmath$ is a retract of $\imath$ and $M = \mathrm{Im}(\imath) \oplus \mathrm{Ker}(\jmath)$ in case (i), respectively s is a section of p and $M = \mathrm{Im}(s) \oplus \mathrm{Ker}(p)$ in case (ii). And if (i), (ii) are satisfied, $(\mathcal{E})$ is called split. Further, a (long) sequence $\cdots \to M_{i-1} \xrightarrow{\phi_i} M_i \xrightarrow{\phi_{i+1}} M_{i+1} \to \dots$ of R-modules is exact at M_i iff $0 \to \mathrm{Im}(\phi_i) \to M_i \to \mathrm{Im}(\phi_{i+1}) \to 0$ is exact (WHY), and similarly for split at M_i (HOW).

Finally, recall that an R-module P is called projective, if P has the *lifting property,* i.e., if for any surjective morphism $f : M \to M''$ of R-modules, $\mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, M'')$ is surjective. Make sure that you know/checked the characterizations of P being projective, e.g. P projective iff $\mathcal{H}_P = \mathrm{Hom}_R(P, \bullet)$ is exact iff P is direct summand in an R-free module...

9) Let M, M_i be R-modules, S be a commutative f.f. R-algebra. Prove/disprove/answer:

a) M is projective iff $M_\mathfrak{p}$ is projective $\forall\, \mathfrak{p} \in \mathrm{Spec}(R)$ iff $M_\mathfrak{m}$ is projective $\forall\, \mathfrak{m} \in \mathrm{Max}(R)$.

b) $M = \oplus_i M_i$ is projective iff all M_i are projective. Does the same hold for $M = \prod_i M_i$?

c) If M is a projective, then M is flat. Does the converse hold/if M is finite R-module?

d) M is projective iff M_S is so. Further, M_1, M_2 are projective iff $M_1 \otimes_R M_2$ is projective.