Due: March 27, 2024

# Math 6030 / Problem Set 7 (one page)

**Special classes of commutative rings**

**1)** Let $R$ be a commutative ring, $\Sigma \subset R$ be a multiplicative system, $R_\Sigma = \Sigma^{-1}R$. Prove/disprove:

a) If $R$ is a PID or a UFD, then $R_\Sigma$ is a PID, respectively a UFD.

b) If $R$ is Noetherian/Artin ring, then $R_\Sigma$ is Noetherian/Artin ring.

c) If $R$ is Euclidean, then $R_\Sigma$ is Euclidean.

**2)** Let $R = \mathbb{Z}[\alpha]$ and $p \in \mathbb{N}$ be prime numbers. Prove/disprove/answer the following:

a) $R$ is Euclidean in each of the cases (i) $\alpha^2 = -1$; (ii) $\alpha^2 = \pm 2$; (iii) $\alpha^2 = \pm 3$.

c) Which $p \in \mathbb{N}$ are prime elements of $R$ in each of the cases (i), (ii), (iii) ?

**3)** Let $f : R \to S$ be a surjective morphism of commutative rings. Prove/disprove:

a) $R$ is a PID, then $S$ is a PID. b) If $R$ is a UFD, then $S$ is a UFD.

**4)** Which of the following is a PID/UFD/Noetherian/Artin/valuation ring ?

a) $R = F[t_1, \dots, t_d]_\mathfrak{p}$, where $F$ is a field and $\mathfrak{p} = (t_1, \dots, t_r)$ for some $r \leqslant d$.

b) $R = \mathbb{Z}[t]_\mathfrak{p}$, where $\mathfrak{p} = (t-1)$, respectively $\mathfrak{p} = (p)$ with $p$ a prime number.

c) $R = \mathbb{Z}[t_1, t_2]/(t_1^2 t_2^3 - 3, t_1^2 - t_2^4)$.

• Recall that a domain $R$ is called integrally closed, if for every $x \in K = \mathrm{Quot}(R)$ one has: If there are $n \geqslant 1$ and $a_0, \dots, a_{n-1} \in R$ s.t. $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0_K$, then $x \in R$.

**5)** Let $R$ be domain, $K = \mathrm{Quot}(R)$ be its field of fractions. Prove/disprove:

a) If $R$ is a UFD, e.g. a PID, then $R$ is integrally closed.

b) Every valuation ring $R$ is integrally closed.

c) Let $R$ be domain, $K = \mathrm{Quot}(R)$, and $\mathrm{Val}_R(K) := \{R_v \in \mathrm{Val}(K) \mid R \subset R_v\}$. Prove:

**Theorem** (Charact. of integrally closed). *$R$ is integrally closed iff* $R = \bigcap_v R_v,\ R_v \in \mathrm{Val}_R(K)$.

**6)** Prove the assertions from the class:

a) Every nontrivial valuation of $\mathbb{Q}$ is equivalent to a unique $p$-adic valuation of $\mathbb{Q}$.

b) Every valuation ring of $F(t)$ is of the form $F[t]_\mathfrak{p}$ with $\mathfrak{p} \in \mathrm{Spec}(F[t])$ of $F[\frac{1}{t}]_{(\frac{1}{t})}$.

**7)** Let $R$ be a commutative ring with $1_R$. Prove/disprove the following:

a) Let $R$ be Noetherian and $f = \sum_n a_n t^n \in R[[t]]$. Then $f$ is nilpotent iff all $a_n$ are so.

b) If $R[t_1, \dots, t_n]$ is Noetherian, then $R$ is Noetherian.

**8)** Let $K$ be a field, $R \subset K$ be a valuation ring, $\mathfrak{p} \in \mathrm{Spec}(R)$, $\mathrm{pr}_R : R \to K_0 := R/\mathfrak{m}$, and $R_0 \subset K_0$ be a valuation ring with valuation ideal $\mathfrak{m}_0$. Prove/disprove:

a) $\mathrm{Spec}(R) \to \{R' \mid R \subset R' \subset K \text{ subring}\}$, $\mathfrak{p} \mapsto R_\mathfrak{p}$ is bijective & $R_\mathfrak{p}$ is val ring with $\mathfrak{m}_{R_\mathfrak{p}} = \mathfrak{p}$.

b) $R_1 := \mathrm{pr}^{-1}(R_0) \subset R$ is a valuation ring of $K$ with $\mathfrak{m}_{R_1} = \mathrm{pr}^{-1}(\mathfrak{m}_0)$ s.t. $R_0/\mathfrak{m}_0 = R_1/\mathfrak{m}_1$ and $0 \to \Gamma_{R_0} \to \Gamma_{R_1} \to \Gamma_R \to 0$ is an exact sequence of totally ordered abelian groups.