4) To every set \( f : X \to Y, \) \( g : Y \to Z \) be a maps of sets.
   a) Show that if \( f \) and \( g \) are injective, then so is \( g \circ f \).
   b) Formulate the converse assertion to a), and prove or disprove it.
   c) Show that if \( f \) and \( g \) are surjective, then so is \( g \circ f \).
   d) Formulate the converse assertion to c), and prove or disprove it.
   e) Show that if \( f \) and \( g \) are bijective, then so is \( g \circ f \).
   f) Formulate the converse assertion to a), and prove or disprove it.

2) Let \( X, Y \) be arbitrary sets, and \( \mathcal{P}(X) \), respectively \( \mathcal{P}(Y) \) denote their power sets. Let further \( f : X \to Y \) be a mapping of sets. Define maps as follows:

\[
f_* : \mathcal{P}(X) \to \mathcal{P}(Y), \quad f^* : \mathcal{P}(Y) \to \mathcal{P}(X)
\]

by \( f_*(A) := f(A) \) the image of \( A \subseteq X \) by \( f \); respectively \( f^*(B) := f^{-1}(B) \) the preimage of \( B \subseteq Y \) under \( f \).

Prove or disprove each of the following assertions:

a) \( f \) is injective \iff \( f_* \) is injective. Respectively: \( f \) is surjective \iff \( f_* \) is surjective.

b) The same assertions as at a) above, but for \( f^* \) instead of \( f_* \).

c) \( f \) is bijective \iff \( f^* \) is bijective. Respectively: \( f \) is bijective \iff \( f_* \) is bijective.

d) \( f^* \) is compatible w. intersection, i.e. \( f^*(A \cap B) = f^*(A) \cap f^*(B) \) for all \( A, B \subseteq Y \).

   Respectively: \( f \) is compatible w. union, i.e. \( f^*(A \cup B) = f^*(A) \cup f^*(B) \)

   e) \( f_* \) is compatible w. intersection, i.e. \( f_*(A \cap B) = f_*(A) \cap f_*(B) \).

   Respectively: \( f \) is compatible w. union, i.e. \( f_*(A \cup B) = f_*(A) \cup f_*(B) \) for all \( A, B \subseteq Y \).

3) Let \( X \) be an arbitrary set, and \( \mathcal{P}(X) \) its power set. For every \( A \in \mathcal{P}(X) \), we define a map \( \chi_A : X \to \{0, 1\} \) by \( \chi_A(x) = 1 \) if \( x \in A \), and \( \chi_A(x) = 0 \) if \( x \notin A \). The function \( \chi_A \) is called the characteristic function of \( A \).

   Prove the following:

   a) Let \( \mathcal{F}_X \) be the set of all the functions from \( X \) to \( \{0, 1\} \). Then the map \( f : \mathcal{P}(X) \to \mathcal{F}_X \) defined by

\[
A \mapsto \chi_A
\]

   is compatible. Respectively:

b) The characteristic functions have the property: For all \( A, B \subseteq X \) one has: \( \chi_{A \cap B} = \chi_A \cdot \chi_B \), and

\[
\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B},
\]

   where “+” and “−” are the usual addition and multiplication in \( \mathbb{N} \).

4) To every set \( X \) we attach the symbol \( |X| \), called the cardinality of \( X \). (Intuitively, \( |X| \) is a kind of set theoretic “size” of \( X \).) We say by definition that \( |X| \leq |Y| \) if there exist injective maps \( f : X \to Y \). And we will say that \( |X| < |Y| \) if the following holds: \( |X| \leq |Y| \) and \( |Y| \neq |X| \). Prove or disprove the following:

a) If \( |X| \leq |Y| \) and \( |Y| \leq |Z| \), then \( |X| \leq |Z| \).

b) \( |X| \leq |Y| \) \iff there exist surjective maps \( g : Y \to X \).

   And \( |X| \leq |Y| \) and \( |Y| \leq |X| \) \iff there exist bijective maps \( f : X \to Y \).

c) For every set \( X \) one has: \( |X| < |\mathcal{P}(X)| \).

Logical deduction

5) Recall the definition of the limit of a function at a point. And recall that we say that function \( f : D \to \mathbb{R} \) on some open interval \( D \) is continuous at \( a \in D \) if \( \lim_{x \to a} f(x) = f(a) \). Show the following:

a) If \( \lim_{x \to a} f(x) = \mu \) and \( \lim_{x \to a} g(x) = \nu \), then \( \lim_{x \to a} \left( f(x)/g(x) \right) = \mu/\nu \), provided \( \nu \neq 0 \).

b) \( f \) is continuous at \( a \) \iff the following holds: Given any open interval \( J \) with \( f(a) \in J \), its preimage \( f^{-1}(J) \) contains some open interval \( I \) with \( a \in I \).