Math 360 - Advanced Calculus / Problem Set 4

Logical deduction

1) Let \((x_n)\) be a Cauchy sequence of real numbers. Let \(k_0 < k_1 < \ldots < k_n < \ldots\) be a strictly increasing sequence in \(\mathbb{N}\), and \((x_{k_n})\) the corresponding sub-sequence of \((x_n)\). Prove or disprove the following:
   a) \((x_{k_n})\) is a Cauchy sequence.
   b) If the answer to a) is positive, then \((x_n)\) and \((x_{k_n})\) are equivalent, i.e., \(\lim (x_n - x_{k_n}) = 0\).

2) Let \((x_n)\) and \((y_n)\) be sequences of real numbers such that \(y_n \leq x_n\) for all \(n\). Prove the following:
   a) **Comparison principle:** If \((x_n)\) and \((y_n)\) are convergent, then \(\lim y_n \leq \lim x_n\).
   b) **“Sandwich” principle:** If \((z_n)\) is a sequence such that \(y_n \leq z_n \leq x_n\) for all \(n\), and \(\lim y_n = \lim x_n\), then \((z_n)\) is convergent, and \(\lim z_n = \lim x_n\).

3) Recall that for equivalence classes of Cauchy sequences \(x = (x_n)/\sim\), \(y = (y_n)/\sim\), we say that \(x \leq y\) if \(\exists N\) such that \(x_n < \epsilon + y_n\) for all \(n > N\).
   a) Is the above definition of \(x \leq y\) equivalent to the following: \(\forall \epsilon > 0 \exists N\) s.t. \(x_n \leq \epsilon + y_n\) for all \(n > N\)?
   b) Prove or disprove: \(x < y \iff \exists \delta > 0 \exists N\) such that \(x_n + \delta < y_n\) for all \(n > N\).

Series

Let \((x_n)\) be a sequence of real numbers. To it we associate the symbol \(\sum_n x_n\) and call it the series defined by \((x_n)\), and say that \(x_n\) the \(n^{th}\) term of the series \(\sum_n x_n\). To a series \(\sum_n x_n\) we attach its sequence of partial sums \((s_n)\), which is defined by \(s_n := \sum_{k=0}^n x_k\).

**Definition.** Let a series \(\sum_n x_n\) be given, and \((s_n)\) be its sequence of partial sums. We say that \(\sum_n x_n\) is convergent, if \((s_n)\) is convergent. If so, we say that \(\sum_n x_n = \lim s_n\) is the value (or the limit) of \(\sum_n x_n\). We further say that \(\sum_n x_n\) is absolutely convergent, if the series \(\sum_n |x_n|\) is convergent. A series \(\sum_n x_n\) is called divergent, if \(\sum_n x_n\) not convergent.

Here, \(|x_n|\) is the absolute value of \(x_n\). What is the definition of \(|x|\) for a real number \(x \in \mathbb{R}\)?

4) The geometric series defined by \(x \in \mathbb{R}\) is \(\sum_n x^n\). (What is \((x_n)\) in this case?) Prove or disprove the following: \(\sum_n x^n\) is convergent \(\iff\) \(\sum_n x^n\) is absolutely convergent \(\iff\) \(-1 < x < 1\).

5) The zeta series at \(s \in \mathbb{R}\) is defined to be the series \(\sum_n n^{-s}\). (What is the sequence defining this series?) Prove or disprove: \(\sum_n n^{-s}\) is convergent \(\iff\) \(\sum_n n^{-s}\) is absolutely convergent \(\iff\) \(s > 1\).

6) Prove or disprove: If \(\sum_n x_n\) is convergent, then \(\lim x_n = 0\). Is the converse true? (Examples?)

7) Prove or disprove: If \(\sum_n x_n\) is absolutely convergent, then \(\sum_n x_n\) is convergent. Is the converse true?

8) To simplify notations and language, one replaces “\(\exists N\) such that \(\forall n > N\)” by “\(\forall n \gg 0\)”.

Prove the following criteria (or tests) for convergence of a series:
   a) **Comparison test:** If \(\sum_n a_n\) is convergent, and \(\forall n \gg 0\) one has \(|x_n| \leq a_n\), then \(\sum_n x_n\) is (absolutely) convergent.
   b) **Ratio test:** If \(\exists r < 1\) such that \(\forall n \gg 0\) one has \(|a_{n+1}/a_n| < r\), then \(\sum_n x_n\) is (absolutely) convergent.
   c) **Limit comparison test:** If \(\sum_n y_n\) is convergent, and \(lim x_n/y_n\) exists, then \(\sum_n x_n\) is convergent.
   d) **Leibniz test:** If \((x_n)\) monotone decreasing, and \(\lim x_n = 0\), then \(\sum (-1)^n x_n\) is convergent.
   e) **Integral test:** If \(f : [a, \infty) \to [0, \infty)\), is decreasing and \(\int_a^\infty f(t)\,dt\) exists, and \(\forall n \gg 0\) one has \(|x_n| \leq f(n)\), then \(\sum_n x_n\) is (absolutely) convergent.

9) Give thoughts to how to turn each of the above convergence tests into a divergence test.