Math 360 - Advanced Calculus / Problem Set 7

Homeomorphisms: Recall that a continuous function \( f : X \to Y \) is said to be a homeomorphism, if there exists a continuous function \( g : Y \to X \) such that \( f \circ g = \text{id}_Y \) and \( g \circ f = \text{id}_X \). In other words, \( f : X \to Y \) is a homeomorphism if \( f \) is a continuous bijection, and its inverse map \( f^{-1} : Y \to X \) is continuous.

Two topological spaces \( X \) and \( Y \) are called homeomorphic, if there exists some homeomorphism \( f : X \to Y \).

1) Let \( f : X \to Y \) be a continuous function. Show that the following are equivalent:
   i) \( f \) is a homeomorphism.
   ii) \( f \) is bijective, and for every open subset \( U \subseteq X \) one has: \( f(U) \) is open in \( Y \).
   iii) \( f \) is bijective, and for every closed subset \( A \subseteq X \) one has: \( f(A) \) is closed in \( Y \).

2) Answer the following:
   a) Give an example of a continuous bijective function \( f : X \to Y \) such that \( f \) is not a homeomorphism.
   b) Let \( f : X \to Y \) be a bijective continuous function. Suppose that \( X \) is compact, and \( Y \) is Hausdorff.
      Show that \( f \) is a homeomorphism.

3) Let \( f : U \to V \) be a continuous bijective function of topological spaces. Prove or disprove:
   a) If \( U, V \subseteq \mathbb{R} \) are open subsets, then \( f : U \to V \) is a homeomorphism.
   b) Is the same true if we replace \( V \) by an arbitrary metric space?
   c) Is the same true if we replace \( U \) by an arbitrary metric space?

4) Prove or disprove the following:
   a) The open interval \( I = (-1, 1) \) is homeomorphic to \( \mathbb{R} \).
   b) A subspace \( X \subseteq \mathbb{R} \) is homeomorphic to \( \mathbb{R} \) if and only if \( X \) is an open interval.

5) Prove or disprove the following:
   a) The open cube \( K = I \times I = \{(x, y) \mid -1 < x, y < 1\} \) is homeomorphic to \( \mathbb{R}^2 \).
   b) The closed cube \( \overline{K} = [-1, 1] \times [-1, 1] = \{(x, y) \mid -1 \leq x, y \leq 1\} \) is homeomorphic to \( \mathbb{R}^2 \).
   c) What are the generalizations of the assertions above for \( \mathbb{R}^n \)?

6) Google the term “space-filling curve” and learn about that. Prove or disprove: The interval \( I = (-1, 1) \) is not homeomorphic to the open cube \( K = I \times I \). How do you explain your answer versus space-filling curves?

Topological equivalence: Recall that two distance functions \( d', d'' \) on a set are called topologically equivalent, if they define the same topology on that set.

7) Let \( d', d'' \) be distances on a set \( X \). Show the following:
   a) \( d', d'' \) are topologically equivalent \( \iff \forall x \in X \forall \epsilon > 0 \exists \delta > 0 \text{ s.t.}: B_{d'}(x, \delta) \subseteq B_{d''}(x, \epsilon) \) and \( B_{d'}(x, \delta) \supseteq B_{d''}(x, \epsilon) \).
   b) Suppose that there exit positive real numbers \( \alpha \leq \beta \) such that \( \alpha \cdot d'(x, y) \leq d''(x, y) \leq \beta \cdot d'(x, y) \) for all \( x, y \in X \). Show that \( d', d'' \) are topologically equivalent.

[Hint: Use the definition of the topology defined by a distance function.]

8) Answer the following:
   a) Draw the unit balls in \( \mathbb{R}^n \), for \( n = 1, 2, 3 \), for the distances \( d_{\text{max}} \), the Euclidean distance \( d \), and the distance \( d_{\text{sum}} \).
   b) Make an educated guess: Are the distances \( d_{\text{max}} \), the Euclidean distance \( d \), and the distance \( d_{\text{sum}} \) topologically equivalent?