Math 360 - Advanced Calculus / Problem Set 9

Let $X, d$ be a metric space. For non-empty subsets $A, B \subseteq X$ we define:

- $d(A, B) = \inf\{d(x, y) \mid x \in A, y \in B\}$, and call it the distance between $A$ and $B$.
- $\delta(A) := \sup\{d(x', x'') \mid x', x'' \in A\}$, and call it the diameter of the set $A$.

1) Let $\mathbb{R}^n$, $n = 1, 2, 3$, be endowed with any of the usual three norms: $\| \cdot \|_1$, and the Euclidean $\| \cdot \|_E$, and the norm $\| \cdot \|_{\sup}$.
   a) What is the diameter of $B(x, r)$ and of $\overline{B}(x, r)$ in every of the norms above?
   b) What is the distance between two balls $B(x', r')$ and $B(x'', r'')$?
   c) Is the same true in arbitrary metric spaces?

2) For $A, B \subseteq X$, let $\overline{A}, \overline{B}$ be their closures. Prove or disprove the following:
   a) $d(A, B) = d(\overline{A}, \overline{B})$.
   b) $\delta(A) = \delta(\overline{A})$.

3) For every metric space $(X, d)$, we endow $X \times X$ with the product topology. Prove the following:
   a) The distance map $d : X \times X \to \mathbb{R}$ is continuous.
   b) The norm map $\| \cdot \| : V \to \mathbb{R}$ is continuous for every normed real vector space $(V, \| \cdot \|)$.
   c) The inner product $<\cdot, \cdot> : V \times V \to \mathbb{R}$ of a real vector space $V$ is continuous.

4) Let $X, d$ be a metric space, and $A, B \subseteq X$ non-empty closed subsets. Prove or disprove the following:
   a) $A \cap B = \emptyset \iff d(A, B) > 0$.
   b) Same question if $A$ and $B$ are compact.

5) In which of the following normed real vector spaces is the unit ball compact?
   a) $\mathbb{R}^n$ for $n > 5$.
      Respectively, in all the $l_q$ with $q \leq 3$.
   b) In all $C_b(X, \mathbb{R})$, provided the space $X$ is finite.
      Respectively, in all $C_0(X, \mathbb{R})$, provided the space $X$ is compact.

6) Let $(V, \| \cdot \|)$ and $(V', \| \cdot \|')$ be normed real vector spaces, and $f : V \to W$ a $\mathbb{R}$-linear map. Prove the following:
   a) $V$ is a Banach space $\iff 0_V$ has a neighborhood $U$ which is complete.
   b) $f$ is continuous $\iff \exists r > 0$ such that $f(B(0, r)) \subseteq B(0, 1)$.
   c) Are there injective continuous $\mathbb{R}$-linear maps $\varphi : l_q \to l_p$ for $p \neq q$?

7) Let $\| \cdot \|, \| \cdot \|'$ be norms on the real vector space $V$. Prove or disprove: $\| \cdot \|$ and $\| \cdot \|'$ define the same topology on $V$ $\iff \exists \alpha, \beta > 0$ such that $\alpha\|v\| \leq \|v\|' \leq \beta\|v\|$ for all $v \in V$.

8) Let $(V', \| \cdot \|')$ and $(V'', \| \cdot \|''')$ be normed real vector spaces, and let $\mathcal{L}(V', V'')$ be the set of all the continuous $\mathbb{R}$-linear maps $\varphi : V' \to V''$. Show the following:
   a) $\mathcal{L}(V', V'')$ endowed with the addition of maps is a real vector space.
   b) For $\varphi \in \mathcal{L}(V', V'')$, the set $\Sigma_{\varphi} := \{\|\varphi(v')\|'' \mid \|v\|' \leq 1\} \subseteq \mathbb{R}$ is bounded.
   c) The map $\| \cdot \| : \mathcal{L}(V', V'') \to \mathbb{R}$ defined by $\|\varphi\| := \sup(\Sigma_{\varphi})$ is a norm.