IMPORTANT:
Points for each problem: 4pt for part a) / 6pt for part b) / 7pt for part a) and b)

1) Let A, B ⊆ \mathbb{R}^n be non-empty subsets, and recall that A + B := \{v + w \mid v \in A, w \in B\}. Prove or disprove:
   a) A and B are bounded \iff A + B is bounded.
   b) If A and B are compact, then A + B is compact.

2) Let z_0, z_1 be fixed points in \mathbb{C}. Define f : \mathbb{R} \to \mathbb{R} by f(x) := \min\{|x - z_0|, |x - z_1|\}. Prove or disprove:
   a) If z_0 = 1, z_1 = i, then f is continuous on \mathbb{R}.
   b) f is differentiable on \mathbb{R} \iff z_0 = z_1 =: z and z is not a real number.

3) Prove or disprove and justify your answer: For every compact and connected subspace A \subset \mathbb{R}^2 one has:
   a) \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} \text{ such that } (x, y) \in A\} is a closed bounded interval in \mathbb{R}.
   b) \{t \in \mathbb{R} \mid \exists (x, y) \in A \text{ such that } t^3 = \exp(x) + y^2\} is a closed bounded interval in \mathbb{R}.

4) Recall that \mathcal{C}_b(X, \mathbb{R}) is the real vector space of the bounded continuous functions on X, and \| \|_\infty is the sup-norm on \mathcal{C}_b(X, \mathbb{R}). Prove or disprove:
   a) Let X = [-1, 1], and f_n : X \to \mathbb{R}, be defined by f_n(x) = \sum_{k=1}^{n} \frac{x^k}{k!}, for all n ≥ 1. Then:
      i) f_n \in \mathcal{C}_b(X, \mathbb{R}) for all n.
      ii) (f_n(x))_n is convergent in \mathbb{R} for all x \in X.
      iii) (f_n)_n is a bounded sequence in \mathcal{C}_b(X, \mathbb{R}).
   b) Same questions for X = \mathbb{R}, and f_n : X \to \mathbb{R}, defined by f_n(x) = \sum_{k=1}^{n} \frac{1}{k^2 + x^2}.

5) Is there a differentiable function f : (-1, 1) \to \mathbb{R} such that
   a) f'(x) < 0 for x < 0, and f'(x) > 0 for 0 < x and f'(0) ≠ 0?
   b) f'(-1, 1) := \{f'(x) \mid -1 < x < 1\} = I \cup J with I, J non-empty disjoint intervals?

6) In which of the following normed real vector spaces is the closed unit ball compact? Justify your answer!
   a) \mathbb{R}^n for 9 < n < 100. Respectively, in I_{2}.
   b) In all \mathcal{C}(I, \mathbb{R}), where I is a properly chosen interval I \subseteq \mathbb{R}.

7) Find the Taylor polynomials of degree n \in \mathbb{N} at a = -1 of the function f : (-\infty, 0) \to \mathbb{R}, where:
   a) f(x) = 1/(1 - x).
   b) f(x) = \log(1 - x).

8) Let X, d be a metric space. Recall that f : X \to \mathbb{R} is called uniformly continuous, if \forall \epsilon > 0 \exists \delta > 0 such that for all x', x'' \in X one has: d(x', x'') < \delta implies |f(x') - f(x'')| < \epsilon. Prove or disprove that in the cases below, f is uniformly continuous:
   a) f : [0, \infty) \to \mathbb{R}, defined by f(x) = \sqrt{x + 10}.
   b) f : [0, \infty) \to \mathbb{R} is continuous, and twice differentiable on (0, \infty), and satisfies f'(x) > 0 and f''(x) < 0 for all x > 0.