Math 360 - Advanced Calculus / Problem Set 4

Real numbers
Recall that given non-empty subsets \( X, Y \subset \mathbb{R} \), we defined \( X + Y := \{ x + y \mid x \in X, y \in Y \} \), and call \( X + Y \) “the sum” of \( X \) and \( Y \); and \( X \cdot Y := \{ xy \mid x \in X, y \in Y \} \), and call \( X \cdot Y \) “the product” of \( X \) and \( Y \).

1) Answer the questions below:
   a) Describe the following sums of sets: \([0, 1] + \mathbb{Z}\), \((0, \infty) + (-\infty, 0)\), \([a, b] + \mathbb{Q}\), \([-2, 1] + [0, 3)\), where \([a, b]\) and \([a, b)\), etc., denote intervals.
   b) The same problem for \( X + Y \) replaced by \( X \cdot Y \).

2) Let \( I \subset \mathbb{R} \) be the set of the irrational numbers. Prove or disprove:
   a) \( a \in I \iff \frac{1}{2}a - 3 \in I; \ b + 1 \in I \iff b^2 - 1 \notin \mathbb{Q} \).
   b) If \( a^5 + 1 = \frac{1}{4} \), then \( a \in I \).
   c) If \( y + z \in I \) and \( yz \in I \), then \( y, z \in I \).

3) Prove or disprove the following:
   a) \( X \) and \( Y \) are both bounded \( \iff \) \( X + Y \) is bounded.
   b) sup \( X \) and sup \( Y \) do both exist \( \iff \) sup \( (X + Y) \) does exist. What is the relation between these numbers, if they all exist.
   c) max \( X \) and max \( Y \) do both exist \( \iff \) max \( (X + Y) \) does exist. What is the relation between these numbers, if they all exist.
   d) \( X \) and \( Y \) are (bounded) intervals \( \iff \) \( X + Y \) is a (bounded) interval.

4) The questions as at Problem 2 above for \( X + Y \) replaced by \( X \cdot Y \).

Sequences
In the problems 5), 6), 7) find the values of \( a \) for which the resulting sequence \((x_n)\) is:
   a) Monotone.
   b) Bounded.
   c) Cauchy, respectively convergent.

5) For a given (rational, or real) number \( a \), define the sequence \((x_n)\) by: \( x_0 := a \), and \( x_{n+1} = x_n^2 - x_n + 1 \) for all \( n \geq 0 \).

6) For a given (rational, or real) number \( a \), define the sequence \((x_n)\) by: \( x_0 := a \), and \( x_{n+1} = \sqrt{|a^2 - x_n^2|} \) for all \( n \geq 0 \).

7) For a given (rational, or real) number \( a \), define the sequence \((x_n)\) by: \( x_0 := a \), and \( x_{n+1} = \sqrt{|2 - x_n^2|} \) for all \( n \geq 0 \).

8) Recall the following definitions:
   - The harmonic series is the symbol \( \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \)
   - The alternating harmonic series is the symbol \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \ldots \)
   a) What could be the meaning of the above symbols?
   b) Define \( \sigma_n := \sum_{k=1}^{n} \frac{1}{k} \). Prove or disprove: \((\sigma_n)\) is a Cauchy sequence.
   c) Define \( \rho_n := \sum_{k=1}^{n} (-1)^k \frac{1}{k} \). Prove or disprove: \((\rho_n)\) is a Cauchy sequence.