Math 360 - Advanced Calculus / Problem Set 10

Homeomorphisms
1) Let \( f : X \to Y \) be a continuous function. Show that the following are equivalent:
   i) \( f \) is a homeomorphism.
   ii) \( f \) is bijective, and for every open subset \( U \subseteq X \) one has: \( f(U) \) is open in \( Y \).
   iii) \( f \) is bijective, and for every closed subset \( A \subseteq X \) one has: \( f(A) \) is closed in \( Y \).
2) Let \( f : X \to Y \) be a bijective continuous function. Suppose that \( X \) is compact, and \( Y \) is Hausdorff. Show that \( f \) is a homeomorphism.
3) Let \( I, J \subseteq \mathbb{R} \) are intervals, and \( f : I \to J \) be a continuous bijective function. Prove or disprove:
   a) \( f : I \to J \) is a homeomorphism.
   b) \( I \) is (half) open/closed \( \iff \) \( J \) is so.
   c) \( f \) is strictly monotone.
4) Prove or disprove the following:
   a) The open interval \( I = (-1, 1) \) is homeomorphic to \( \mathbb{R} \).
   b) More general, any two open non-empty intervals \( I, J \subseteq \mathbb{R} \) are homeomorphic.
   c) A subspace \( X \subseteq \mathbb{R} \) is homeomorphic to \( \mathbb{R} \) \( \iff \) \( X \) is an open interval.
5) Prove or disprove the following:
   a) The open cube \( K = I \times I = \{(x, y) \mid -1 < x, y < 1\} \) is homeomorphic to \( \mathbb{R}^2 \).
   b) Every open cube \( I \times J \), with \( I, J \subseteq \mathbb{R} \) open non-empty intervals, is homeomorphic to \( \mathbb{R}^2 \).
   c) What are the generalizations of the assertions above for \( \mathbb{R}^n \)?
6) Google the term “space-filling curve” and learn about that. Prove or disprove: The interval \( I = (-1, 1) \) is not homeomorphic to the open cube \( K = I \times I \). How do you explain your answer versus space-filling curves?

Metric spaces
7) Let \( X', d' \) and \( X'', d'' \) be metric spaces. we set set \( X := X' \times X'' \). Prove or disprove that the following maps on \( X \times X \) are distances:
   a) \( d_1 : X \times X \to \mathbb{R} \), by \( d_1((x', x''), (y', y'')) = d'(x', y') + d''(x'', y'') \).
   b) \( d_E : X \times X \to \mathbb{R} \), by \( d_E((x', x''), (y', y'')) = \sqrt{d'(x', y')^2 + d''(x'', y'')^2} \).
   c) \( d_{\text{max}} : X \times X \to \mathbb{R} \), by \( d_{\text{max}}((x', x''), (y', y'')) = \max\{d'(x', y'), d''(x'', y'')\} \).
8) Answer the following:
   a) Draw the unit balls in \( \mathbb{R}^n \), for \( n = 1, 2, 3 \), for the distances \( d_1 \), the Euclidean distance \( d_E \), and the distance \( d_{\text{max}} \).
   b) Make an educated guess: Do the distance map \( d_1 \), the Euclidean distance \( d_E \), and the distance \( d_{\text{max}} \) define the same topology on \( \mathbb{R}^n \)?