Math 360 - Advanced Calculus / Problem Set 11

Metric spaces
1) Let $X, d$ be a metric space. Prove or disprove the following:
   a) Let $A \subseteq X$ non-empty. Then $x \in \overline{A} \Leftrightarrow \exists (x_n)_n$ with $x_n \in A$ and $\lim_{n \to \infty} x_n = x$.
   b) Suppose that $X$ is complete. Then $A$ is closed $\Leftrightarrow A$ is complete.
2) Let $I = [0, 1]$, and $V \subset C(I, \mathbb{R})$ be the subset defined by: $f \in V \Leftrightarrow \exists \epsilon_f > 0$ s.t. $f(x) = 0$ for $0 \leq x \leq \epsilon_f$.
   And for $n > 1$ consider $f_n : I \to \mathbb{R}$ defined by $f(x) = 0$ for $0 \leq x \leq \frac{1}{n}$ and $f_n(x) = \frac{n}{n-1}x - \frac{1}{n-1}$ for $\frac{1}{n} \leq x \leq 1$. Show/answer the following:
   a) $V$ is a real vector subspace of $C(I, \mathbb{R})$.
   b) $f_n \in V$ for all $n > 1$.
   c) $(f_n)_n$ is a Cauchy sequence in the sup-norm $\| \cdot \|_\infty$.
   d) $(f_n)_n$ has no limit in $V$.
   e) $(f_n)_n$ has a limit in $C(I, \mathbb{R})$. What is $\lim_{n \to \infty} f_n$?
3) Which of the following sets are bounded, respectively connected, respectively compact:
   a) $\{(x, y) \in \mathbb{R}^2 \mid x^2 < 3, |y| \leq 2\}$
   b) $\{v \in \mathbb{R}^n \mid d_{\| \cdot \|_\infty}(v, 0) \leq 10\}$.
   c) $A \subset \mathbb{C}$ finite non-empty set.
   d) The boundary of a bounded non-empty set $A \subset \mathbb{R}^n$.
   e) $A = \{(a, b) \in \mathbb{R}^2 \mid a \in \mathbb{Q}\}$.
   f) The unit ball in $l_1$.
4) Prove the assertion made in the class: If $X$ is a topological space, and $A \subset X$ is a path-connected subset, then $A$ is connected.
5) Prove or disprove the following assertions:
   a) If $A \subset \mathbb{R}^n$ is connected, then $\mathbb{R}^n \setminus A$ is disconnected.
   b) If $A \subset \mathbb{R}^n$ is compact, then $\mathbb{R}^n \setminus A$ is not compact.
   c) If $A \subset \mathbb{R}^n$ and $\mathbb{R}^n \setminus A$ are both connected, then one of them must be bounded.
   e) Every open and connected subset of $\mathbb{R}^n$ is path-connected.
6) Let $V$ endowed with $\| \cdot \|$ be a normed vector space. For non-empty subsets $A, B \subseteq V$ we denote their sum by $A + B = \{v + w \mid v \in A, b \in B\}$. Prove or disprove the following:
   a) $A, B$ are bounded iff $A + B$ is bounded.
   b) $A, B$ are compact iff $A + B$ is compact.
   c) $A, B$ are connected iff $A + B$ is connected.

Supplement: The completion of a metric space.
Let $X, d$ be a metric space, and let $C(X)$ be the set of all the Cauchy sequences $f : \mathbb{N} \to X$ with values in $X$. (Recall that for such an $f : \mathbb{N} \to X$, we usually write $(x_n)_n$ with $x_n := f(n)$. In order to simplify notations, we will though prefer to denote a sequence by $f$, understanding that that is a map from $\mathbb{N}$ to $X$.)
7) Define $\delta : C(X) \times C(X) \to \mathbb{R}$ by $\delta(f, g) = \lim_{n \to \infty} d(f(n), g(n))$. Prove the following:
   a) $\delta$ is well defined, i.e., $\lim_{n \to \infty} d(f(n), g(n))$ exists for all $f, g \in C(X)$.
   b) Show that $\delta$ is symmetric, i.e., $\delta(f, g) = \delta(g, f)$ for all $f, g \in C(X)$.
c) Show that $\delta$ is satisfies the triangle inequality, i.e., $\delta(f, h) \leq \delta(f, g) + \delta(g, h)$ for all $f, g, h \in \mathcal{C}(X)$.

d) Show that if $\delta(f, f') = 0$ and $\delta(g, g') = 0$, then $\delta(f, g) = \delta(f', g')$ for all $f, f', g, g' \in \mathcal{C}(X)$.

8) For every $f \in \mathcal{C}(X)$, define $\hat{f} := \{ f' \mid \delta(f, f') = 0 \}$. And let $\hat{X} = \{ \hat{f} \mid f \in \mathcal{C}(X) \}$. This is a set of subsets of $\mathcal{C}(X)$.

Define $\hat{d} : \hat{X} \times \hat{X} \to \mathbb{R}$ by $\hat{d}(\hat{f}, \hat{g}) := \delta(f, g)$. Using the previous Problem, prove the following:

a) For all $f, g$ one has: If $\hat{f} \cap \hat{g}$ is not empty, then $\hat{f} = \hat{g}$, hence $\delta(f, g) = 0$.

b) The map $\hat{d} : \hat{X} \times \hat{X} \to \mathbb{R}$ is well defined, and is a distance map.

c) $\hat{X}$ is complete w.r.t. the distance map $\hat{d}$.

9) Finally, define $i : X \to \hat{X}$, $x \mapsto f_x$ the constant Cauchy sequence $f_x : \mathbb{N} \to X$, by $f(n) = x$. Prove the following:

a) $i$ is an isometric embedding, i.e., $d(x, y) = \hat{d}(\hat{f}_x, \hat{f}_y)$ for all $x, y \in X$.

b) $i(X)$ is a dense subset of $\hat{X}$.

**Language:** One says that $i : X \to \hat{X}$ is the completion of $X, d$. 
