Math 371 / Problem Set 3

Products of groups/rings/fields

In the sequel $(G_1, \cdot)$, $(G_2, \cdot)$ are groups, and $G = G_1 \times G_2$ is their product endowed with the composition law as defined in class $(x_1, x_2) \ast (y_1, y_2) := (x_1y_1, x_2y_2)$. Recall that $(G, \ast)$ is a group.

1) Prove or disprove the following:
   a) $G$ is abelian iff both $G_1$ and $G_2$ are abelian.
   b) $G$ is simple iff both $G_1$ and $G_2$ are simple.

2) Let $H_1 \subseteq G_1$, $H_2 \subseteq G_2$ be subsets, and set $H := H_1 \times H_2 \subseteq G_1 \times G_2 = G$ viewed as subset of $G$. Prove or disprove / answer the following:
   a) $H$ is a (normal) subgroup of $G$ iff both $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ so.
   b) How does $G/H$ relate to $G_1/H_1$ and $G_2/H_2$?

In the sequel $(R_1, +, \cdot)$, $(R_2, +, \cdot)$ are rings, and $R = R_1 \times R_2$ is endowed with the coordinate wise addition: $(x_1, x_2) + (y_1, y_2) := (x_1 + y_1, x_2 + y_2)$ and multiplication: $(x_1, x_2) \cdot (y_1, y_2) := (x_1y_1, x_2y_2)$.

3) Prove or disprove the following:
   a) $(R, +, \cdot)$ is a ring, and $R$ is a (commutative) ring [with $1_R$] iff both $R_1$ and $R_2$ are so, respectively.
   b) One has $R^\times = R_1^\times \times R_2^\times$, and $(x, y) \in R$ is a zero divisor iff $x \in R_1$ and $y \in R_2$ are so.

4) Prove or disprove the following: $R$ is a (skew) field iff $R_1$ and $R_2$ are so.

Groups/rings of functions

In the sequel, $(G, \cdot)$ is a group, and $(R, +, \cdot)$ is a ring. For a non-empty set $X$, let $\mathcal{F}(X, G)$, respectively $\mathcal{F}(X, R)$, denote all the functions from $X$ to $G$, respectively $R$. Further, for a non-empty subset $Y \subseteq X$, let $\phi : \mathcal{F}(X, ?) \to \mathcal{F}(Y, ?)$ defined by $f \mapsto g$ with $g(x) = f(x)$ for all $x \in Y$ be the restriction map.

5) Define a composition law $\ast$ on $\mathcal{F}(X, G)$ by $(f \ast g)(x) := f(x) \cdot g(x)$ for all $f, g \in \mathcal{F}(X, G)$. Prove or disprove the following:
   a) $\mathcal{F}(X, G)$ endowed with $\ast$ is group, and $\ast$ is commutative iff $\cdot$ is so.
   b) $\phi : \mathcal{F}(X, G) \to \mathcal{F}(Y, G)$ is a group homomorphism.
   c) If the answer at b) is positive, give a description of Ker$(\phi)$ and Im$(\phi)$.

6) Let $H \subseteq G$ be a subset, and $\mathcal{F}(X, H)$ be all the maps from $X$ to $H$. We view every map $f : X \to H$ as a map $f : X \to H \subseteq G$, hence we have an inclusion $\mathcal{F}(X, H) \subseteq \mathcal{F}(X, G)$ (WHY?). Prove or disprove / answer the following:
   a) $H$ is a subgroup of $G$ iff $\mathcal{F}(X, H)$ is a subgroup $\mathcal{F}(X, G)$.
   b) $H$ is a normal subgroup of $G$ iff $\mathcal{F}(X, H)$ is a normal subgroup $\mathcal{F}(X, G)$.
   c) Comment on the relation between $\mathcal{F}(X, G/H)$ and $\mathcal{F}(X, G)/\mathcal{F}(X, H)$.

7) Let us endow $\mathcal{F}(X, R)$ with an addition defined by $(f \oplus g)(x) := f(x) + g(x)$, and a multiplication defined by $(f \odot g)(x) := f(x) \cdot g(x)$ for all $x \in X$ and $f, g \in \mathcal{F}(X, R)$. Prove or disprove / answer the following:
   a) $(\mathcal{F}(X, R), \oplus, \odot)$ is a ring, which is commutative iff $R$ is commutative, and has identity iff $R$ has identity.
   b) Describe the group of units $\mathcal{F}(X, R)^\times$.
   c) Describe the zero divisors of $\mathcal{F}(X, R)$.

8) Prove or disprove: $\mathcal{F}(X, R)$ is a (skew) field iff $|X| = 1$ and $R$ is a (skew) field.