Math 371 / Problem Set 5

Modules and vector spaces (continued)

1) Let $M$ be an $R$-module, and $N \subseteq M$ be an $R$-submodule. We set $\overline{M} := M/N$ and denote its elements by $\overline{x} := x + N$ for all $x \in M$. Suppose that $x_1, x_2 \in N$ are generators of $N$, and let $y_1, y_2 \in M$ be some fixed elements. Prove or disprove the following:
   a) If $(x_1, x_2, y_1, y_2)$ is a systems of generators of $M$, then so is $(\overline{y}_1, \overline{y}_2)$ for $\overline{M}$.
   b) If $\overline{y}_1, \overline{y}_2$ generate $\overline{M}$, then $x_1, x_2, y_1, y_2$ generate $M$.

2) Prove or disprove the following:
   a) Let $F$ be a field. Then $(F, +)$ and $(F^2, +)$ are not isomorphic as $F$-vector spaces.
   b) Is the same true for $(R, +)$ and $(R^2, +)$ viewed as $R$-modules, where $R$ is a ring with $1_R$?
   c) Is the same true if we replace $F^2$ by $F^n$ and $R^2$ by $R^n$ with $n > 1$?

3) Let $f : M \rightarrow N$ be an $R$-module homomorphism. Let $X \subseteq M$ be a non-empty subset, and $Y = f(X)$ be its image in $N$. Prove or disprove the following:
   a) If $x = \sum_i a_i x_i$ is an $R$-linear combination of elements of $X$, then $f(x)$ is an $R$-linear combination of elements of $Y$.
   b) $f(\langle X \rangle_R) = \langle Y \rangle_R$.
   c) If $M$ is finitely generated, then so is $\text{Im}(f)$.

4) Let $M$ be an $R$-module, and $\Sigma$ be a non-empty set. Let $\mathcal{F}(\Sigma, M)$ be the set of all the maps $f : \Sigma \rightarrow M$. Then $\mathcal{F}(\Sigma, M)$ is an abelian group w.r.t. the addition of functions (WHY?), and we define an outer $R$-multiplication on $\mathcal{F}(\Sigma, M)$ by: $(rf) : \Sigma \rightarrow M, x \mapsto rf(x)$, for all $r \in R$, and $f, g \in \mathcal{F}(\Sigma, M)$.
   a) Show that in this way, $\mathcal{F}(\Sigma, M)$ becomes an $R$-module.
   b) Show that $\mathcal{F}(\Sigma, M)$ is a finitely generated $R$-module iff $\Sigma$ is finite.

5) Let $F$ be a field, and $\mathcal{P}ol_n \subseteq F[X]$ be the set of all the polynomials of degree $\leq n$. Prove or disprove the following:
   a) $\mathcal{P}ol_n$ is an $F$-vector subspace of $(F[X], +)$.
   b) $(1_F, \ldots, 1_F + \ldots + X^n)$ is an $F$-basis of $\mathcal{P}ol_n$.
   c) For every finitely generated $F$-vector subspace $V \subseteq F[X]$, there exists $n$ such that $V \subseteq \mathcal{P}ol_n$.

6) Which of the above assertions remain true if we replace $F$ and $F[X]$ by an arbitrary ring $R$, respectively $R[X]$, and $V \subseteq F[X]$ a finitely generated $R$-submodule $M$ of $(R[X], +)$.

7) Let $R$ be an arbitrary ring with $1_R$.
   a) Prove that for all $n > 0$ the following $R$-modules are isomorphic:
      i) $R^n$ with the coordinate wise addition.
      ii) $\mathcal{P}ol_{n-1}$ with the usual addition of polynomials.
      iii) $\mathcal{F}(\Sigma, R)$, where $\Sigma$ is a finite set with $|\Sigma| = n$.
   b) Is the same true if $R$ has no $1_R$?

8) Let $M, N$ be $R$-modules, and let $\text{Hom}_R(M, N)$ be the set of all the $R$-module homomorphisms $f : M \rightarrow N$, hence $\text{Hom}_R(M, N) \subseteq \mathcal{F}(M, N)$ as a set. Prove or disprove:
   a) $\text{Hom}_R(M, N)$ is an $R$-submodule of $\mathcal{F}(M, N)$.
   b) Suppose that $(x_1, \ldots, x_n)$ is a system of generators of $M$. Then for $f, g \in \text{Hom}_R(M, N)$ one has: $f = g$ iff $f(x_i) = g(x_i)$, $i = 1, \ldots, n$.
   c) Suppose that $M$ and $N$ are finitely generated (having $m$, respectively $n$, generators). Then $\text{Hom}_R(M, N)$ is a finitely generated $R$-module (having $nm$ generators).