Math 371 / Problem Set 6

1) Let $M$ be an $R$-module, and recall that for a subset $T \subseteq M$, we denote by $<T>_R$ the $R$-submodule of $M$ generated by $T$. Let $X, Y \subseteq M$ be subsets. Prove or disprove the following:
   a) $< <X>_R>_R = <X>_R$.
   b) If $X \subseteq Y$, then $<X>_R \subseteq <Y>_R$. Is the converse true?
   c) $<X>_R + <X>_R = <X \cup Y>_R$.

2) Let $M$ be an $R$-module. An element $x \in M$ is called a torsion element if $rx = 0$ for some $r \neq 0$. Let $M_{\text{tors}} := \{x \in M \mid x$ is torsion element $\}$. Prove or disprove the following:
   a) $M_{\text{tors}}$ is an $R$-submodule of $M$.
   From now on, suppose that $R$ be an integral domain.
   b) Let $x \in M$ be such that $rx \in M_{\text{tors}}$ for some $r \neq 0$. Then $x \in M_{\text{tors}}$.
   c) Let $\overline{M} := M/M_{\text{tors}}$. Then $(\overline{M})_{\text{tors}} = \{0\}$.

3) Answer the following:
   a) What is the torsion submodule for the $\mathbb{Z}$-modules $(\mathbb{Z}, +)$, $(\mathbb{Z}/35\mathbb{Z}, +)$?
   b) Suppose that $R$ is an integral domain, and $M, N$ are $R$-modules. Is then $(M \times N)_{\text{tors}} = M_{\text{tors}} \times N_{\text{tors}}$? Is the same true if $R$ is not a domain?

- Problems 2–9 at the end of Section 4.2, Chapter 4, of Herstein’s book *Topics in Algebra*. 