Math 502 / Problem Set 3

Recall that for a group $G$ and a subgroup $H$, we denote by $G/H$ and $H \setminus G$ the set of left, respectively right, cosets of $G$ with respect to $H$, and we say that $|H\backslash G| =: (G : H) := |G/H|$ is the index of $H$ in $G$.

1) Let $G$ be a group, and $H \subseteq G$ a subgroup. Prove or disprove:
   a) Suppose that $|G/H| = 2$. Then $H$ is normal in $G$.
   b) Suppose that $|G/H| = n$, and that $H$ is the only subgroup of $G$ of index $n$. Then $H$ is normal in $G$.

2) Let $G$ be a group, and $H \subseteq G$ a subgroup. We set $H_{0} := \cap_{x \in G} xHx^{-1}$. Prove or disprove:
   a) $xHx^{-1}$ is a subgroup of $G$ for all $x \in G$, and if $x \in H$, then $xHx^{-1} = H$.
   b) $H_{0}$ is a normal subgroup of $G$.
   c) $H_{0}$ is the unique maximal normal subgroup of $G$ which is contained in $H$.
   d) If $(G : H)$ is finite, then $(G : H_{0})$ is finite.
   e) Can one give an upper bound for $(G : H_{0})$ in terms of $(G : H)$?

3) Let $(G_{1}, \cdot), (G_{2}, \cdot)$ be groups, and $G = G_{1} \times G_{2}$ is their product endowed with the component wise multiplication $(x_{1}, x_{2}) \cdot (y_{1}, y_{2}) := (x_{1}y_{1}, x_{2}y_{2})$. Let $H_{1} \subseteq G_{1}, H_{2} \subseteq G_{2}$ be subsets, and set $H := H_{1} \times H_{2}$ viewed as subset of $G = G_{1} \times G_{2}$. Prove or disprove / answer the following:
   a) $G$ is an (abelian) group iff both $G_{1}$ and $G_{2}$ are so.
   b) $G$ is simple iff both $G_{1}$ and $G_{2}$ are simple.
   c) $H$ is a (normal) subgroup of $G$ iff both $H_{1} \subseteq G_{1}$ are $H_{2} \subseteq G_{2}$ so.
   d) How does $G/H$ relate to $G_{1}/H_{1}$ and $G_{2}/H_{2}$?

4) Let $1 \to G' \xrightarrow{f} G \xrightarrow{g} G'' \to 1$ be an exact sequence of groups. Prove or disprove the following:
   a) $G$ is finitely generated iff $G'$ and $G''$ are so.
   b) Suppose that $G''$ is abelian. Then a subgroup $H \subseteq G$ is normal iff $H \cap f(G')$ is normal in $G$.
   c) Let $H \triangleleft G$ be such that $g$ maps $H$ isomorphically onto $G''$. Then $G$ is isomorphic to $G' \times G''$.

[Hint: Prove and use the following: First, $H' := H$ and $H'' := f(G')$ satisfy: $H', H'' \triangleleft G$, $H' \cap H'' = \{e_{G}\}$, and $H'H'' = G$. Second, $G/H' \cong H'', G/H'' \cong H'$, and the canonical homomorphism $G \to G/H' \times G/H''$, $g \mapsto (gH', gH'')$ is an isomorphism.]

• Problems 1–8 at the end of §6, Ch.1, of Hungeford’s *Algebra* book.