Math 512 / Problem Set 9

1) Let $V$ be an $F$-vector space, and $p(X) ∈ F[X]$ be a non-zero polynomial over $F$. Let $\phi ∈ \text{End}_F(V)$ be an $F$-endomorphism of $V$, and $\psi := p(\phi)$ viewed as $F$-endomorphism of $V$.

a) Prove or disprove: If $\lambda$ is an eigenvalue of $\phi$, then $\mu := p(\lambda)$ is an eigenvalue of $\psi$.

b) Does the converse of assertion a) hold?

2) Let $V$ be an $F$-vector space, and $\phi, \psi ∈ \text{End}_F(V)$ be $F$-endomorphisms of $V$. Prove or disprove the following:

a) There exists an $F$-basis $A$ of $V$ such that both $A^A_φ$ and $A^A_ψ$ are diagonal iff $\phi \circ \psi = \psi \circ \phi$.

b) What is the corresponding assertion for finitely many $F$-endomorphisms $φ_1, \ldots, φ_n$ of $V$?

Language: If there exists $A$ as above, one says that $\phi$ and $ψ$ are simultaneously diagonalizable.

3) Give thoughts to possible generalizations of the above assertions for $F$-endomorphisms $\phi$ of $V$ whose characteristic polynomial $P_φ(X)$ splits in linear factors over $F$.

4) Consider all the matrices over $F$ whose characteristic polynomial is $P_A(X) = (X − 2)^2(X − 3)^3$.

a) Give all the possible Jordan normal forms for these matrices (a possible hint to problem 3 above?).

b) Does the characteristic of the field $F$ play into the picture?

Language: Recall that for every polynomial $p(X) = a_nX^n + a_{n−1}X^{n−1} + \cdots + a_1X + a_0$ (over an arbitrary commutative ring $R$ with $1_R$) one defines the formal derivative $p'(X)$ of $p(X)$ as being the $0$ if $n = 0$, i.e., $p(X) = a_0$, and being the polynomial $p'(X) = na_nX^{n−1} + (n−1)a_{n−1}X^{n−2} + \cdots + a_1$, if $n > 0$.

5) Let $D : \text{Pol}_3 → \text{Pol}_3$ be the $F$-linear map defined by the formal derivative, and $A := (1, X, X^2, X^3)$ be the standard $F$-basis of $\text{Pol}_3$.

a) What is the matrix of $D$ in the $F$-basis $A$?

b) What is the Jordan normal form of $D$?

Language: If there exists $A$ as above, one says that $D$ is simultaneously diagonalizable.

6) Let $F = \mathbb{C}$ be the field of complex numbers. Find the Jordan normal form of all the matrices $A$ satisfying:

a) $A^2 = A$, respectively $A^3 = A$.

b) $A^n = I_n$ and $A ∈ F^{n×n}$.

Language: Recall that $\phi ∈ \text{End}_F(V)$ is called nilpotent, if there exists some $n$ such that $\phi^n = 0$ in $\text{End}_F(V)$.

7) Prove or disprove / answer the following:

a) $\phi$ is nilpotent iff $P_φ(X) = X^{\dim(V)}$.

b) What are the possible Jordan normal forms for nilpotent $F$-endomorphisms?

8)* Let $φ ∈ \text{End}_F(V)$ be an endomorphism, and $P_φ(X)$ its characteristic polynomial. Prove or disprove the following: $P_φ(X)$ splits in linear factors over $F$ if and only if there exists $φ_1, φ_0 ∈ \text{End}_F(V)$ such that $φ_1 \circ φ_0 = φ_0 \circ φ_1$, and $φ = φ_0 + φ_1$, and $φ_0$ is diagonalizable and $φ_0$ is nilpotent. If so, are $φ_0, φ_1$ unique?

Language: If $φ_0$ and $φ_1$ as above exist, then $φ = φ_0 + φ_1$ is called the canonical decomposition of $φ$. 