Math 621 (Number Theory II) / Problem Set 2

1) Let $K$ be a locally compact non-discrete topological field. Show that its topology is a $V$ topology.

   [Hint: Let $U = \{x \in K \mid x^n \to 0\}$. Then $U$ is open, thus $U^{-1}$ is open too. Let $S$ be its complement. Show that $S$ is compact. (Start with some compact neighborhood $\Sigma$ of 0, and look at $\cup a^n\Sigma, a \in \mathbb{Z}$.) Etc.]

2) Continuity of the roots: Let $K$ be a complete field with respect to the absolute value $|\cdot|$. Let $x \in K$ a simple root of $f(X) \in K[X]$, and $U \subset K$ a neighborhood of $x$. Show that if $g(X) \in K[X]$ is coefficient-wise close enough to $f(X)$, then $g(X)$ has a simple root $y \in U$.

   Witt vectors:

3) Fix a prime number $p$. For a system of variables $T = (T_m)_m$ define a system $W(T) = (W_n(T))_n$ of polynomials from $\mathbb{Z}[T]$ by $W_n(T) = \sum_0^n p^mT_m^{p^n - m}$. Set $X = (X_m)_m$ and $Y = (Y_m)_m$. Show the following:

   a) There exist systems of polynomials $S = (S_m)_m$ and $P = (P_m)_m$ in $\mathbb{Z}[X,Y]$ such that we have identities:

      \[ W(S) = W(X) + W(Y) \quad \text{and} \quad W(P) = W(X) \cdot W(Y). \]

   b) Let $R$ be any commutative Ring with $1$. For $a = (a_m)_m$ and $b = (b_m)_m$ from $W(R) := R^{\mathbb{N}}$ define:

      \[ a + b = (S_0(a,b), S_1(a,b), \ldots) \quad \text{and} \quad a \cdot b = (P_0(a,b), P_1(a,b), \ldots) \]

      Show that the above $+$ and $\cdot$ define a structure of a commutative ring on $W(R)$.

   c) What is this ring if $p$ is invertible in $R$?

   Language: $W(R)$ is called the ring of Witt vectors over $R$.

4) In the above context, suppose that $k$ is a perfect field having $\operatorname{char}(k) = p$. Define on $W(k)$ the Verschiebung $V$ by $V(a) = (0, a_0, a_1, \ldots)$ and the Frobenius $F$ by $F(a) = (a_p^0, a_p^1, \ldots)$. Show the following:

   a) $V$ is an endomorphism of $(W(k), +)$.

   b) $F$ is a ring automorphism of $W(k)$.

   c) $V \circ F = F \circ V = p \operatorname{id}_{W(k)}$.

5) (Continued) Show the following:

   a) If $k$ is a perfect field with $\operatorname{char}(k) = p$, then $W(k)$ is a complete DVR, $pW(k)$ is its maximal ideal, and $k$ is its residue field. Describe the Teichmüller system of representatives $\overline{\pi} : k \to W(k)$.

   b) Let $R$ be a complete DVR such that its maximal ideal is generated by $p$, and its residue field $k$ is perfect with $\operatorname{char}(k) = p$. Then $R$ is canonically isomorphic to $W(k)$.

   c) Moreover, if $l$ is another perfect field, then one has a canonical bijection $\operatorname{Hom}(k, l) \to \operatorname{Hom}(W(k), W(l))$.

Higher ramification groups

6) Describe the higher ramification groups at the primes over 2 and 3 in $K/\mathbb{Q}$, where

   a) $K = \mathbb{Q}((\zeta_3, \alpha), \alpha^3 = 2$  

   b) $K = \mathbb{Q}((\zeta_3, \alpha), \alpha^3 = 3$  

7) The same question for the $n^{\text{th}}$ cyclotomic field $\mathbb{Q}(\zeta_n)/\mathbb{Q}$, where $n = p^e$ for some $e \geq 1$. 