1) Let \( K \) be an arbitrary field. For \( n \geq 1 \) we denote by \( \text{GL}_n(k) \) and \( \text{SL}_n(k) \) the group of invertible \( n \times n \) matrices, respectively the ones with determinant 1.

a) For each \( n \) and \( k \) prove or disprove: \( \{1\} \triangleleft \{\pm 1\} \triangleleft \text{SL}_n(k) \triangleleft \text{GL}_n(k) \) is a normal series of \( \text{GL}_n(k) \). In the cases where the answer is positive, determine the factors \( F_i \).

b) For each of these factors \( F_i \) prove or disprove: \( F_i \) is a simple group.

2) Let \( G \) be a finite group, and \( N, M \) be normal subgroups of \( G \) such that \( N \cap M = \{1\} \). Show the following:

a) \( M \) and \( N \) commute element-wise, i.e., \( gh = hg \) for all \( g \in N, h \in M \).

b) In particular, if \( N, M \) together generate \( G \), then \( G \cong N \times M \) canonically.

3) Let \( G \) be a finite group, and for every prime number \( p \) we denote by \( S_p(G) \) a Sylow \( p \)-group of \( G \). (In particular, if \( p \) does not divide the order of \( G \), then by definition we set \( S_p(G) = \{1\} \).) Prove the following assertions made/used in the class:

a) \( S_p(G) \) is a normal subgroup of \( G \) iff \( G \) has a unique Sylow \( p \)-group.

b) \( G \cong \prod_p S_p(G) \) if and only if \( G \) has a unique Sylow \( p \)-group for each \( p \).

4) Let \( G \) be a finite group. Prove or disprove:

a) \( G \) has a unique maximal solvable quotient \( G_{\text{sol}} \).

b) \( G \) has a unique maximal solvable normal subgroup \( G_{\text{sol}} \).

5) Let \( G \) be a finite group. Prove or disprove:

a) \( G \) has a unique maximal nilpotent quotient \( G_{\text{nil}} \).

b) \( G \) has a unique maximal nilpotent normal subgroup \( G_{\text{nil}} \).

Q: Does one get the same answers at 4) and 5) for arbitrary groups \( G \)?

6) An element \( g \) of a group \( G \) is called a non-generator, if for every set \( S \subset G \) one has: If \( S \) does not generate \( G \), then \( S \cup \{g\} \) does not generate \( G \). For a finite group \( G \) prove the following:

a) \( g \in G \) is a non-generator iff \( g \in M \) for all the maximal subgroups \( M \) of \( G \).

Definition The group \( \Phi(G) = \cap M \), where \( M \) runs through all the maximal subgroups of \( G \), is called the Frattini subgroup of \( G \). Thus the set of the non-generators of \( G \) is exactly the Frattini subgroup \( \Phi(G) \) of \( G \).

b) Show that \( \Phi(G) \) is normal in \( G \). What is \( \Phi(G/\Phi(G)) \)?

c) Prove or disprove: \( \Phi(G) \) is a characteristic subgroup of \( G \).

7) Let \( G \) have order \( (G : 1) = pq \), with \( p, q \) prime numbers. Prove or disprove (in dependence of the prime numbers \( p \) and \( q \)):

a) \( G \) is solvable.

b) \( G \) is nilpotent.

c) \( G \) is Abelian, respectively cyclic.

Do the answers change radically if \( (G : 1) = pqr \), with \( p, q, r \) prime numbers?

8) Let \( p \) be a prime number.

a) If \( G \) is a non-commutative \( p \)-group, then \( (G : Z(G)) > p \).

b) Describe the isomorphism types of groups of order \( p^3 \).