1) Problems 2 and 5 from PS 5 revisited:
Let $A$ be any commutative ring with $1_A$. For every integer $k \geq 0$ define $A$-linear maps $\delta^k : A[X] \to A[X]$ such that $\delta^k(X^m) = 0$ for $k > n$, and $\delta^k(X^n) = \binom{n}{k}X^{n-k}$ for $k \leq n$. In particular, $\delta^0 = \text{id}_{A[X]}$. Finally set $P^{[k]}(X) = \delta^k(P)$ for all $P(X) \in A[X]$ and $k \geq 0$. (These are called the “formal Hasse–Witt derivatives.) For any $A$-algebra $B$ and $a \in B$, show the following:
   a) $P(X) = \sum_{k \geq 0} P^{[k]}(a)(X - a)^k$ in $B[X]$.
   b) $(X - a)^n$ divides $P(X)$ in $B[X] \iff P^{[k]}(a) = 0$ for all $0 \leq k < n$.
2) Let $\overline{K}|K$ be a field extension with $K$ algebraically closed, and $L|K$ an algebraic extension. Let $\mathcal{S}_{L/K}$ be the set of all the $K$-embeddings of $L$ into $\overline{K}$. Define a map $\text{Aut}_K(L) \times \mathcal{S}_{L/K} \to \mathcal{S}_{L/K}$ by $(g, \psi) \mapsto \psi \circ g^{-1}$. Show the following:
   a) The map above defines an action of $\text{Aut}_K(L)$ on $\mathcal{S}_{L/K}$.
   b) $L|K$ is normal $\iff$ the above action is transitive, i.e., for every $\psi, \eta \in \mathcal{S}_{L/K} \exists g \in \text{Aut}_K(L)$ such that $\eta = \psi \circ g$.
3) Let $K = \mathbb{F}_q$ be a finite field, say $q = p^n$.
   a) Show that every algebraic extension $L|K$ is Galois.
   b) Suppose that $L|K$ is finite, say $[L : K] = d$. Show that the Galois group $G(L|K)$ of $L|K$ is cyclic generated by the $n^{th}$ power of the Frobenius $\text{Frob}^n : L \to L$, $x \mapsto x^{p^n}$. What is the order of $G(L|K)$?
   Hint: Revisit/use Problems 7, 8 from PS 4.
4) Let $K = \mathbb{Q}$. Find the normal hull $L^n|K$ and describe the Galois group $G(L^n|K)$ of $L^n|K$ in the following cases: $L = K[\sqrt{5}], L = K[\sqrt{5}]$, $L = K[\sqrt{5}]$.
5) Same problem for $L = K[\sqrt{d}], L = K[\sqrt{d}], L = K[\sqrt{d}]$, where $d$ is an arbitrary integer. If the answer depends on $d$, indicate how does it depend on $d$.
6) Same problem in the case $K = \mathbb{R}(t)$ is the rational function field in the variable $t$ over $\mathbb{R}$, and $L = K[\sqrt{7}]$, where $n$ is an arbitrary natural number. Start by first analyzing the cases $n = 2, 3, 4, 5$.
7) Let $K$ be an arbitrary field. Prove or disprove: All field extensions $L|K$ with $[L : K] = 2$ are algebraic normal extensions. And if $\text{char}(K) \neq 2$, then $L|K$ is Galois.
8) Let $K$ be a field with $\text{char}(K) \neq 2$ satisfying the following conditions:
   i) For all $a \in K$, either $a$ or $-a$ is a square in $K$.
   ii) The sum of two squares in $K$ is again a square in $K$.
   a) Show that $L = K[\sqrt{-1}]$ has no extensions of degree 2.
   b) What is a familiar example of such a field $K$? Does there exist such a field $K$ with $K|\mathbb{Q}$ is algebraic, but $K$ not algebraically closed?