Grad Algebra (602) / Problem Set 11

Due: Have fun!!!

Recall that for a group $G$ acting on a set $X$, and an arbitrary group $A$, we denote by $A \rtimes_X G$ the semi-direct product $A^X \rtimes G$ w.r.t. the canonical action of $G$ on $A^X$; and denote $A \rtimes S_n = A^n \rtimes S_n$.

1) Prove or disprove:
   a) The Sylow 2-group $S_2$ of $S_4$ is isomorphic to $(\mathbb{Z}/2) \rtimes S_2$.
   b) For some particular values of $n$—and if so, which are these values, one has: The Sylow 2-group $S_2$ of $S_{2n}$ is isomorphic to the Sylow 2 group of $(\mathbb{Z}/2) \rtimes S_n$.

2) Give examples of Galois extension $L/K$ of $K = \mathbb{Q}$ such that the following holds:
   a) $G(L/K) \cong (\mathbb{Z}/2) \rtimes S_2$.
   b) $G(L/K) \cong (\mathbb{Z}/2) \rtimes S_3$.

3) Let $K$ be a field, and $p$ be a rational prime number such that $p \neq \text{char}(K)$.
   a) Suppose that $p \neq 2$. Prove or disprove: For all $a \in K$ and $P(X) = X^p - a \in K[X]$ one has: $P(X)$ is reducible over $K$ if and only if $P(X)$ has a root in $K$, i.e., $a$ is an $n$th power in $K$.
   b) Describe the situation in the case $p = 2$.

Resolvents. Let $L/K$ be a cyclic extension, and $G = G(L/K)$ be its Galois group. We fix a generator $\sigma$ of $G$, and a normal generator $\alpha$ of $L/K$ such that $\text{Tr}_{L/K}(\alpha) = 1$. (Such normal generators do exist, why?)

a) Suppose that $\text{char}(K)$ does not divide $n = [L : K]$, and that $K$ contains a primitive $n$th root of unity $\zeta$.
   We set $x := \sum_{k=0}^{n-1} \zeta^{-k} \sigma^k(\alpha)$, and call it a Lagrange resolvent of $L/K$.
   b) Suppose that $[L : K] = \text{char}(K) = p$. We set $x := \sum_{k=0}^{p-1} (-k) \sigma^k(\alpha)$, and call it an Artin–Schreier resolvent of $L/K$.

4) Show the following:
   a) In case a) above, one has $\sigma(x) = \zeta x$; and $\text{Mipo}_K(x) = x^n - a$ for some $a \in K$. In other words, $L$ is generated over $K$ by the $n$th root of some element of $K$.
   b) In the case b) above, one has $\sigma(x) = x + 1$; and $\text{Mipo}_K(x) = x^p - x - a$ for some $a \in K$. In other words, $L$ is generated over $K$ by the root of an Artin–Schreier polynomial over $K$.

5) Let $L/K$ be a Galois extension, and $G = G(L/K)$ be its Galois group. For $n \geq 1$ we consider $\mathcal{M}_n(L)$ and $\text{GL}_n(L)$ endowed the $G$-action defined by acting on the coefficients of the matrices in discussion. Prove the following:
   a) The above action $G$ makes both $(\mathcal{M}_n(L), +)$ and $(\text{GL}_n(L), \cdot)$ into $G$-groups; even into topological $G$-groups, when we endow them with the discrete topology.
   b) Show that $H^1(G, \mathcal{M}_n(L)) = 0$ and that $H^1(G, \text{GL}_n(L)) = 0$.

Note: The last assertion is called the Hilbert 90 for $\text{GL}_n$.

6) Let $L/K$ be a finite separable field extension. Let $L^n/K$ be its normal closure (in some fixed algebraic closure of $K$). Prove or disprove:
   a) If $[L : K] = 8$, and $16 \neq [L^n : K] < 32$, then $L^n = L$.
   b) Same question in the case $[L : K] = 12$.

7) Let $L/K$ be an algebraic extension allowing a tower of sub-extensions $K = L_m < \cdots < L_0 = L$ such that $[L_i : L_{i+1}] = 3$ for all $i = 0, \ldots, m - 1$. Prove or disprove:
   a) If each $L_i/L_{i+1}$ is cyclic, and $L^n/K$ is a normal closure of $L/K$, then $L^n/K$ is Galois, and $G(L^n/K)$ is a 3-group.
   b) $\text{Aut}(L^n/K)$ is a 2, 3 group, i.e., its order is not divisible by primes $p > 3$.
   c) $\text{Aut}(L^n/K)$ is solvable.