Grad Algebra (Math 603) / Exam

For each of the rings $A = \mathbb{Z}[X]/(12, X^2)$, $B = \mathbb{R}[X, Y]/(X^2 + Y^2)$ and $C = \mathbb{C}[X]$ do the following:

1) Describe the prime spectrum, and the nil-radical.

2) Describe the maximal spectrum, and the Jacobson radical.

3) Describe the zero divisors and the nilpotent elements.

4) Viewing $A$ and $B$ as $\mathbb{Z}$-algebras, compute $A \otimes_{\mathbb{Z}} B$. And viewing $B$ and $C$ as $\mathbb{R}$-algebras, compute $B \otimes_{\mathbb{R}} C$.

5) Let $R$ be the ring of fractions of $\mathbb{Q}[X]$ w.r.t. the multiplicative system consisting of the powers of $X$.

   a) Prove or disprove: $R$ is a $\mathbb{Q}$-algebra of finite type.

   b) In the case the answer to question a) above is positive, find a Noether basis of $R$ over $\mathbb{Q}$.

6) Set $R = \mathbb{R}[X, Y](X^2 - Y^3) =: \mathbb{R}[x, y]$, where $x = X \pmod{(X^2 - Y^3)}$ and $y = Y \pmod{(X^2 - Y^3)}$. Prove or disprove:

   a) $R$ is an domain, respectively $R$ is integrally closed.

   b) Let $R_0 := \mathbb{R}[t]$ be the polynomial ring in the variable $t$ over $\mathbb{R}$. Then there exists a unique ring homomorphism $f : R_0 \rightarrow R$ such that $t \mapsto y$.

   c) If $f$ as above exists, compute the integral closure of $R_0$ in $R$.

   d) And if $f$ as above exists, describe the fibers of $f^* : \text{Spec}(R) \rightarrow \text{Spec}(R_0)$.

7) In the notations from above, let $\tilde{R}$ be the integral closure of $R$ in its total ring of fractions. Show the following:

   a) $\tilde{R}/R_0$ is a quasi-Galois ring extension with automorphism group $\cong C_2$ the cyclic group of two elements.

   b) For each of primes $p_0 = (t)$, $p_1 = (t - 1)$, $p_2 = (t - 2)$ of $R_0$, find the decomposition group of a prime $q_0, q_1, q_2$ of $\tilde{R}$ lying above, correspondingly.

   c) Describe the invertible prime ideals of $R$, respectively $\tilde{R}$. Are the rings $R$, respectively $\tilde{R}$, Dedekind rings?

8) Let $A$ be a commutative ring with 1, $p$ a prime ideal of $A$, and $\kappa(p)$ the field of fractions of $A/p$. Prove or disprove:

   a) One always has $A_p/p_p = \kappa(p)$.

   b) One always has $A/p = \kappa(p)$.

   c) For which rings $A$ are the assertions a), b) above true for all prime ideals $p \in \text{Spec}(A)$?

   d) For which rings $A$ there do exist prime ideals $p$ such that the assertions a), b) above are true?