Grad Algebra (Math 603) / Problem Set 11 (two pages)

Algebraic independence / Algebras of f.t. over fields

1) Let $K = \mathbb{F}_2$.
   a) Let $K((t)) = \text{Quot}(K[[t]])$ be the function field of the power series ring in the variable $t$ over $K$. Show that $x = \sum_n t^nt^n$ is not algebraic over $K(t)$.
   b) Show that $\text{tr.deg}(K((t))|K)$ is uncountable.
   c) Is the same true for an arbitrary base field $K$?

2) Let $K$ be an arbitrary field.
   a) Find a Noether basis for $R := K[t^{-1}, (t^2 - 1)^{-1}]$, where $t$ is transcendental over $K$.
   b) Same question for $R := K[X_1, X_2, X_3]/(X_1X_2^2 + X_2X_3^3 - 6)$. And find the integral closure of $R$ in its total ring of fractions.

3) Let $R$ be a valuation ring. Prove the following:
   a) Krull.dim($R) \leq 1 \iff \Gamma_R$ is embeddable into $(\mathbb{R}, +)$ as an ordered group.
   b) $R$ is Noetherian $\iff R$ is a PID $\iff R$ is a UFD $\iff \Gamma_R \cong (\mathbb{Z}, +)$.
   c) If Krull.dim($R) > 1$, then $R$ does not satisfy the conclusion of Krull’s Principal Ideal Theorem.

Example: Let $K_0$ be any base field, $K_1 := K_0((t_1))$ the Laurent power series ring over $K_0$ in the variable $t_1$, and $K_2 := K_1((t_2))$ the Laurent power series ring over $K_1$ in the variable $t_2$. Notation: $K_2 := K_0((t_1))((t_2))$. Show that $K_2$ has a canonical valuation ring $R$ with $\Gamma_R \cong \mathbb{Z} \times \mathbb{Z}$ ordered lexicographically and $\kappa_R = K_0$. Thus $R$ is/is not...

Prove the following useful and famous fact:

Finiteness Lemma. Let $A$ be a Noetherian domain, $K = \text{Quot}(A)$ its ring of fractions. Let $L|K$ be a finite field extension, and $B$ the integral closure of $A$ in $L$. Then $B$ is a finite $A$-module, hence Noetherian, in any of the following cases:
   i) $A$ is integrally closed, and $L|K$ is separable.
   ii) $A$ is an algebra of finite type over some field $k$.

4) Prove i) along the following lines:
   a) There exists a $K$-basis $B = (\beta_1, \ldots, \beta_n)$ of $L$ with $\beta_i \in B$ for all $1 \leq i \leq n$.
   b) Let $B^* = (\beta_1^*, \ldots, \beta_n^*)$ be the $K$-basis of $L$ which satisfies Tr$_{L|K}(\beta_i^* \beta_j) = \delta_{ij}$ for all $i, j$. Then one has $B \subseteq A\beta_1^* + \ldots + A\beta_n^*$, hence $B$ is a finite $A$-module.

Hint: Let $x = \sum_i a_i \beta_i$. Then Tr$_{L|K}(x\beta_i) = a_i$ (WHY?). And $x \in B$ implies $a_i \in A$ (WHY?).

5) Prove ii) along the following reduction steps:
   a) Let $T = (t_1, \ldots, t_d)$ be a Noether basis of $A$ over $k$. Then $B$ equals the integral closure of $R_0 := k[T]$ in $L$.
   b) Let $M$ be a finite extension of $L$, and $C$ the integral closure of $A$ in $M$. If $C$ is a finite $A$-module, then so is $B$.
   c) Now set $K_0 = k(T) = \text{Quot}(R_0)$. Then w.l.o.g. we can suppose that $L|K_0$ is normal.
   d) W.l.o.g. we can suppose that $L|K_0$ is pure inseparable.
   e) W.l.o.g. we can suppose that $L|K_0$ is of the form: $L = k_1[T_1]$, where $k_1|k$ is finite pure inseparable, and $T_1 = (u_1, \ldots, u_d)$ with $u_i^p = t_i$ for some power $p^n$ of $p = \text{char}(k)$.
   f) Conclude the proof of ii).
Hint: To b), recall that $B$ is a submodule of $C$, etc. For c) apply b). For d), apply the transitivity of being finite together with Problem 4. To e), remark that if $a^{p^e} = p(t_1, \ldots, t_d)$ for some $p(t_1, \ldots, t_d) \in k[T]$, then $a$ is contained in the extension of $k[T]$ generated by the $(p^e)$th roots from all the coefficients of $p(t_1, \ldots, t_d)$ and of all the $t_i$ (WHY?), etc.

Note: In general, in the above context, the integral closure $B$ of $A$ in $L$ is not a finite $A$-module. An example is the following: $u \in \mathbb{F}_p[[t]]$ be a non-$p$-power in $\mathbb{F}_p[[t]]$ and not algebraic over $\mathbb{F}_p(t)$. Set $A = \mathbb{F}_p(t, u^p) \cap \mathbb{F}_p[[t]]$. Then $A$ is a DVR of $K := \mathbb{F}_p(t, u^p)$ (WHY?). Let $L = \mathbb{F}_p(t, u)$. Then $L|K$ has degree $p$ (WHY?), hence finite. But the integral closure of $A$ in $L$ is not a finite $A$-module (WHY?).

6) Let $K|\mathbb{Q}$ be a number field, and $\mathcal{O}_K$ be its ring of algebraic integers. Show the following:
   a) $\mathcal{O}_K$ is a Dedekind ring.
   b) Let $K = \mathbb{Q}[\sqrt{-5}]$. Compute the representation of $(2), (3), (1 - \sqrt{-5}), (1 + \sqrt{-5})$ as products of prime ideals of $\mathcal{O}_K$.

7) Let $A$ be a domain, $K = \text{Quot}(A)$ its quotient field. Prove the following:
   a) A fractional ideal $M$ of $A$ is invertible $\iff$ $M$ is a projective $A$-module.
   Next suppose that $A$ is a Dedekind ring.
   b) If the ideal class group $\mathcal{C}(A)$ is a torsion group, then every over-ring $A \subseteq B \subseteq K$ of $A$ is a fraction ring of $A$.
   c) Does the converse of the assertion from b) hold?

8) Let $A$ be a Dedekind ring with $K := \text{Quot}(A)$. Show the following:
   a) Every finite non-trivial projective $A$-module $M$ is of the form $M \cong A^{-1} \oplus a$, where $a$ is an ideal of $A$ and $r \geq 1$. Here $r$ is the dimension of $M \otimes_A K$ as $K$-vector space.
   b) If $M \cong A^{-1} \oplus a$ and $M \cong A^{-1} \oplus b$ with $a, b$ ideals of $A$, then $r = s$ and the ideal classes $\hat{a}$ and $\hat{b}$ are equal.
   c) Show that $\mathbb{Z} \oplus \mathcal{C}(A)$ has a ring structure as follows: $(r, \hat{a}) + (s, \hat{b}) := (r + s, \hat{a} \hat{b})$, and $(r, \hat{a}) \cdot (s, \hat{b}) := (rs, \hat{a} \hat{b}^s)$.
   d) Finally, show that the Grothendieck ring $K(A)$ is isomorphic to $\mathbb{Z} \oplus \mathcal{C}(A)$, via $A^{-1} \oplus a \mapsto (r, \hat{a})$.

9) Let $A$ be a domain, and $R$ an $A$-algebra of finite type which has no non-trivial zero-divisors. Prove or disprove the following:
   a) If $A = \mathbb{Z}$, then $R$ is catenary.
   b) If $R = \mathbb{Z}_p$, then $R$ is non necessarily catenary.
   c) What could be a necessary and sufficient condition on $A$ such that all $R$ as above are catenary?

Hint: To a), if $\mathfrak{m}$ is a maximal ideal of $R$, what can you say about $\kappa_{\mathfrak{m}} := R/\mathfrak{m}$? To b), what about ideals of the form $(at - 1) \subseteq \mathbb{Z}_p[t]$?