Problem Set 5

Cyclotomic number fields

1) Let $\mathbb{Q}_n = \mathbb{Q}[\mu_n]$ be the $n^{th}$ cyclotomic number field. Complete the proof of the assertion that $\mathcal{O}_{\mathbb{Q}_n} = \mathbb{Z}[\mu_n]$.
   a) First for prime number power $n = p^a$, then in general.

2) Describe the decomposition/inertia/ramification groups of the cyclotomic number fields $\mathbb{Q}_n | \mathbb{Q}$. Try to do the same for the maximal cyclotomic extension $\mathbb{Q}_{\text{cycl}} | \mathbb{Q}$.

3) Prove the special cases “left to the reader” of the Gauß Reciprocity Law:
   \[
   (\frac{-1}{p}) = (-1)^{(p-1)/2} \quad \text{and} \quad (\frac{2}{p}) = (-1)^{(p^2-1)/8}
   \]

4) For a number field $K | \mathbb{Q}$, let $\text{Div}^{ur}(K)$ be the group of divisors of $\mathcal{O}_K$ which are not ramified over $\mathbb{Z}$. Describe the Artin map $(\cdot, L|\mathbb{Q}) : \text{Div}^{ur}(K) \rightarrow \text{Gal}(K|\mathbb{Q})$ in the following cases:
   a) $K = \mathbb{Q}[\sqrt{d}]$ is a quadratic number field.
   b) $K = \mathbb{Q}[\sqrt{d}, \sqrt{e}]$ is a biquadratic number field.
   c) $K = \mathbb{Q}_n$ is the $n^{th}$ cyclotomic field.

Question: Which problem concerning primes appears to be related to the Artin map in case c)?

Norm and Co-norm (Inclusion)

Let $R$ be a Dedekind ring, $K = \text{Quot}(R)$, $i : K \hookrightarrow L$ a finite field extension such that the integral closure $S$ of $R$ in $L$ is a finite $R$-module. Thus for each $p \in \text{Specmax}(R)$ we have $[L : K] = \sum_{q \in X_p} c(q|p) f(q|p)$. One defines the following group homomorphisms, called co-norm (or inclusion), respectively norm:

\[
\mathcal{I}_{S|R} : \text{Div}(R) \rightarrow \text{Div}(S), \quad p \mapsto \sum_{q \in X_p} c(q|p) q, \quad \mathcal{N}_{S|R} : \text{Div}(S) \rightarrow \text{Div}(R), \quad q \mapsto f(q|p)p, \quad \text{if} \quad q \in X_p.
\]

Finally, let $d_{S|R} : K^\times \rightarrow \text{Div}(R)$ and $d_{S|L} : L^\times \rightarrow \text{Div}(S)$ be the (principal) divisor maps.

5) In the above context show that following:
   a) $\mathcal{I}_{S|R}$ is an embedding of free Abelian groups. Prove or disprove: The norm $\mathcal{N}_{S|R}$ is surjective. Finally show that $\mathcal{N}_{S|R} \circ \mathcal{I}_{S|R}$ is the multiplication by $[L : K]$ on $\text{Div}(R)$.
   b) Functoriality: If $K \hookrightarrow L \hookrightarrow M$ and the resulting $R \hookrightarrow S \hookrightarrow T$ are as above, then the following hold:
      \[
      \mathcal{I}_{T|S} = \mathcal{I}_{T|S} \circ \mathcal{I}_{S|R} \quad \text{and} \quad \mathcal{N}_{T|S} = \mathcal{N}_{S|R} \circ \mathcal{N}_{T|S}.
      \]

6) Show that in the above context, the following hold:
   a) $\mathcal{I}_{L|K}(\text{div}_R(x)) = \text{div}_S(i(x))$.
   b) Show that $\mathcal{N}_{L|K}(\text{div}_S(y)) = \text{div}_R(N_{L|K}(y))$, where $N_{L|K} : L \rightarrow K$ is the usual norm for $L|K$.

Hint to b). Reduce to the case $L|K$ is Galois. Further, by localizing we can suppose that $R = R_p$ is local, thus $S$ is PID.

Due: Friday, Nov 1, 2003