Problem Set 8

1) For a given prime number $p$, consider $\Sigma = \{0, \ldots, p-1\}$ as system of representatives for the elements of $\mathbb{F}_p$. Find the representations of the form $\sum a_i p^i$ with $a_i \in \Sigma$ of the following $p$-adic numbers:
   a) $-1 \in \mathbb{Z}_2$, $2/5 \in \mathbb{Z}_7$.
   b) The $4$th roots of unity in $\mathbb{Z}_5$

2) For each $a$, $0 < a < p$, consider the sequence $(x_n)_n$ in $\mathbb{Q}_p$ with $x_n = a p^n$. Show the following:
   a) $(x_n)_n$ is a Cauchy sequence in $\mathbb{Q}_p$.
   b) Set $\zeta_n = \lim x_n$. Then $\zeta_n$ is a representative for $a$ in $\mathbb{Z}_p$, and $\zeta_n \in \mu_{\mathbb{Q}_p}$.

3) Describe/detect the group of roots of unity of $\mathbb{Q}_p$. Further, for any given $n$, detect the degree $[\mathbb{Q}_p(\mu_n) : \mathbb{Q}_p]$ of the $n$th cyclotomic field of $\mathbb{Q}_p$.

Isomorphisms of local fields

4) Prove or disprove:
   a) $\mathbb{R}$ and $\mathbb{Q}_p$ are not isomorphic as abstract fields. The same for $\mathbb{Q}_p$ and $\mathbb{Q}_q$, $p \neq q$.
   b) Is the same question for arbitrary finite extensions $L$ of $\mathbb{R}$ and $K$ of $\mathbb{Q}_p$, respectively $\mathbb{Q}_q$.
   c) The same question for arbitrary algebraic extensions of $\mathbb{R}$ and $\mathbb{Q}_p$.

Hint. What about $\sqrt{p}$ in any of the fields above, etc.

For the next problems, let $(K, | |)$ be a complete field, and let $| |$ also denote the unique prolongation of $| |$ to a separable/algebraic closure $K^s \subset K^a$ of $K$. Using the Higher Dim Hensel’s Lemma show the following:

5) Krasner’s Lemma: For each $\alpha \in K^s$ set $\Delta(\alpha) = \min |\alpha - \alpha'|$, where $\alpha' \neq \alpha$ are all the conjugates of $\alpha$ in $K^s$. Show that if $\alpha, \beta \in K^s$ such that $|\alpha - \beta| < \Delta(\alpha)$, then $K(\alpha) \subset K(\beta)$.

6) Continuity of the roots: Let $f(X) = a_n X^n + \ldots + a_0 \in K[X]$ with $a_n \neq 0$, with roots $\alpha_i$ having multiplicities $n_i$. Then for each $\epsilon > 0$ there exits $\delta > 0$ such that the following holds: If $g(X) = b_n X^n + \ldots + b_0 \in K[X]$ satisfies $|a_m - b_m| < \delta$ for all $m = 0, \ldots, n$, then $g(X)$ has $n_i$ roots (if counted with their multiplicities) in each ball $B_\epsilon(\alpha_i)$. In particular, if $\epsilon$ is small enough, then each $B_\epsilon(\alpha_i)$ contains exactly $n_i$ roots of $g(X)$ (if counted with their multiplicities).

7) Hensel’s Lemma: Suppose that $| |$ is non-archimedean, say with valuation ring $(\mathcal{O}, m)$, and residue field $\kappa = \mathcal{O}/m$. Let $f(X) \in \mathcal{O}[X]$ be a polynomial, and $\overline{f}(X) \in \kappa[X]$ its image (mod m). Suppose that $\overline{f}(X) = \overline{g}(X) \overline{h}(X)$ in $\kappa(X)$ with $\overline{g}(X)$ and $\overline{h}(X)$ relatively prime, and $\overline{g}(X)$ unitary. Then there exist unique pre-images $g(X), h(X) \in \mathcal{O}[X]$ of $\overline{g}(X)$ and $\overline{h}(X)$ with $g(X)$ unitary, such that $f(X) = g(X) h(X)$. 

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